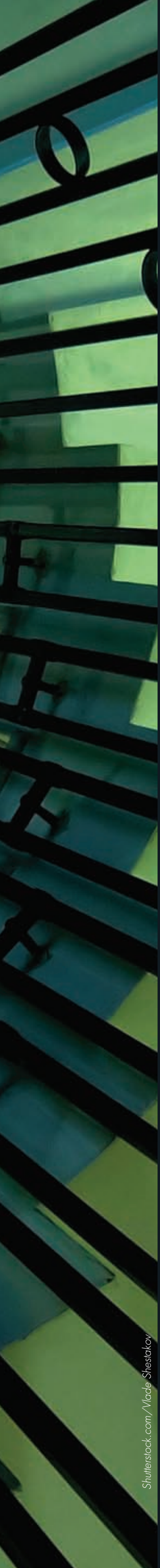


Shutterstock.com/Vlade Shestakov

# 6

## TERMINOLOGY

alternate segment
arc
concyclic
cyclic
intercepts
proof
secant
subtend
tangent

GEOMETRY

# CIRCLE GEOMETRY



Prior learning

## CIRCLE PROPERTIES AND THEIR PROOFS INCLUDING THE FOLLOWING THEOREMS

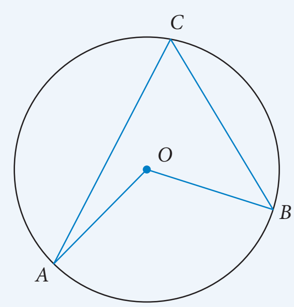
- An angle in a semicircle is a right angle (ACMSM029)
- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc (ACMSM030)
- Angles at the circumference of a circle subtended by the same arc are equal (ACMSM031)
- The opposite angles of a cyclic quadrilateral are supplementary (ACMSM032)
- Chords of equal length subtend equal angles at the centre and conversely chords subtending equal angles at the centre of a circle have the same length (ACMSM033)
- The alternate segment theorem (ACMSM034)
- When two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord (ACMSM035)
- When a secant (meeting the circle at $A$ and $B$) and a tangent (meeting the circle at $T$) are drawn to a circle from an external point $M$, the square of length of the tangent equals the product of the lengths to the circle on the secant. ($AM \times BM = TM^2$) (ACMSM036)
- Suitable converses of some of the above results (ACMSM037)
- Solve problems finding unknown angles and lengths and prove further results using the results listed above. (ACMSM038) AC

# 6.01 ANGLES AT THE CENTRE OF CIRCLES

There are many theorems in Euclidean geometry that relate specifically to circles. The centre of a circle is usually labelled as $O$.

### INVESTIGATION The angle at the centre of a circle

Consider the circle with points $A$, $B$ and $C$ on the circumference, with lines $AO$, $BO$, $AC$ and $BC$ as shown. $\angle ACB$ is on the circumference and $\angle AOB$ is at the centre, as shown in the diagram below.




Construct the segment $CO$ and extend it to point $D$ as shown on the top right.

For convenience, write $\angle CAO = \alpha$ and $\angle CBO = \beta$.

What kind of triangle is $\triangle COB$?

How do you know?

What is $\angle BCO$ equal to?



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How is $\angle DOB$ related to $\angle BCO$ and $\angle CBO$?

What is $\angle DOB$ equal to?

What is $\angle DOA$ equal to?

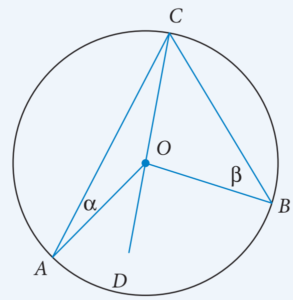
How do you know?

What is $\angle AOB$ equal to?

What is $\angle ACB$ equal to?

What does this prove about $\angle AOB$ and $\angle ACB$?

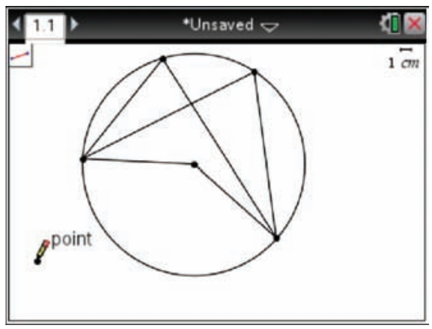
What does this prove about angles subtended at the centre and circumference of a circle by the same arc?



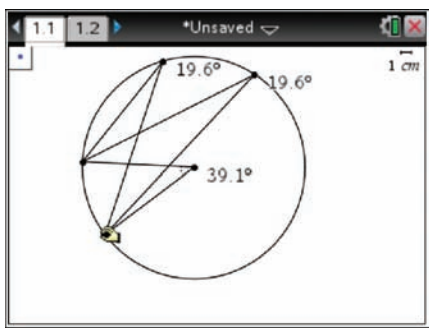

Circle properties can be explored with CAS calculators.

### TI-Nspire CAS

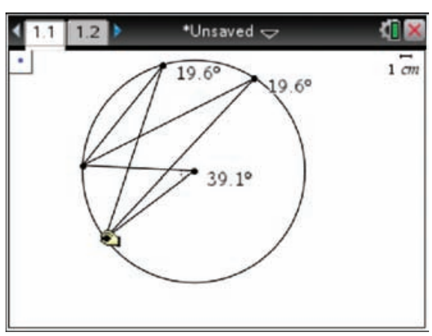
Use a Geometry page.
Use menu, 5: Shapes and 1: Circle to draw a circle.
Use the 'point' to place the centre and move out to place the circumference.
Then use menu, 4: Points & Lines and 5: Segment to draw the lines shown on the screenshot on the right.




Now use menu, 6: Measurement and 4: Angle to measure some of the angles. Move the 'point' to a point on one ray of the angle (even one of the ends of the segment), the centre point of the angle and a point on the other ray of the angle. Notice in the screenshot that the angles have been rounded so they may not be exactly as shown.




Now use menu, 4: Points & Lines and 1: Point. Grab a point on the circle. Move it around the circumference. What do you observe?



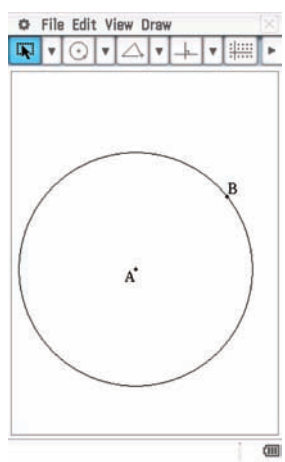

## ClassPad

Use the **Geometry** menu.

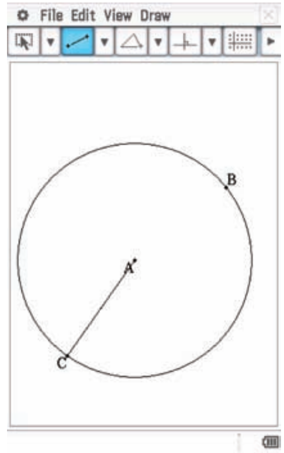

Use the top menu bar to choose the circle drawing tool .

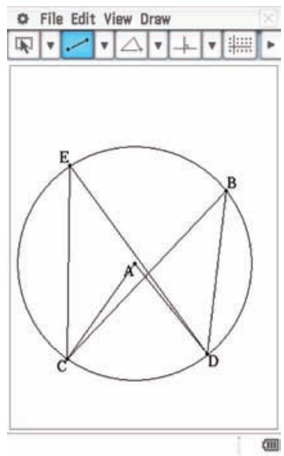

To draw a circle, tap a point in the middle of the screen for the centre and a point further out that will be on the circumference of the circle.

From the menu where you found  choose the line segment tool  to draw the angle subtended at the centre and two angles subtended at the circumference (including one at point *B*) from the same arc.

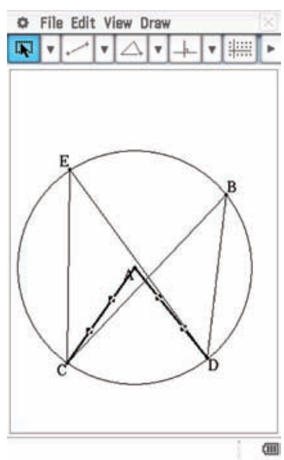


To draw a line from the circumference to the centre, first tap a point on the circumference, then tap point A.

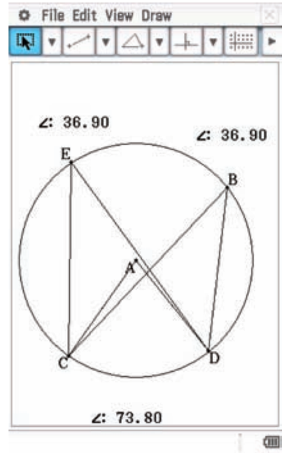



Tap the selection arrow
Select lines *AC* and *AD* for ∠ *CAD* by tapping them.



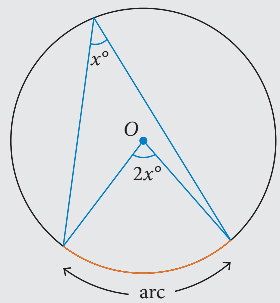
Tap the top right arrow ▸.
Touch and drag the angle measurement below the circle.
Tap the arrow ◂ and repeat the sequence above for ∩ *CED* and ∩ *CBD*, but drag the measurements to the top left and top right of the circle.
Note: Before selecting the next angle, you may need to tap a blank area of screen to remove all previous selections.




## IMPORTANT

### Theorem 1: Angle at the centre of a circle

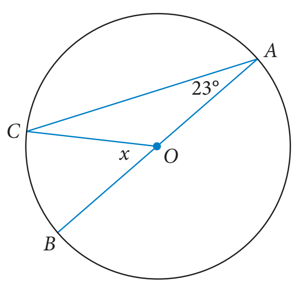
The **angle at the centre** subtended by an arc of a circle is twice the **angle at the circumference** subtended by the same arc.



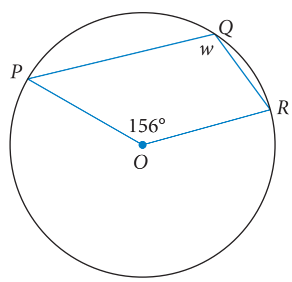

## Example 1

Find the value of the unknowns.

**a**




**b**




### Solution

**a** $\angle CAB$ and $\angle COB$ are both standing on arc *BC*. $x = 2 \times 23°$ (Angle at centre)

Write the answer. $\therefore x = 46°$

**b** Find the reflex ∠*PQR*. | Reflex $\angle POR = 360° - 156°$ (Revolution) $= 204°$

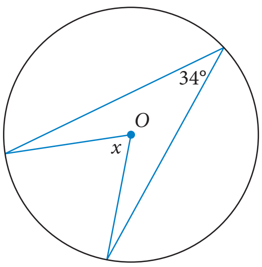
∠ *PQR* and reflex ∠ *POR* are both on the major arc *PR*. | $2w = 204°$ (Angle at centre)

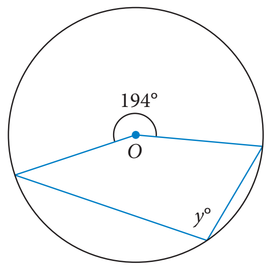
Write the answer. | $w = 102°$

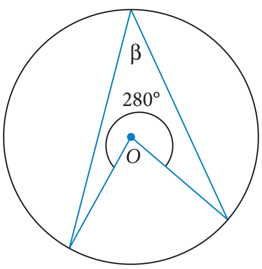
# EXERCISE 6.01 Angles at the centre of circles

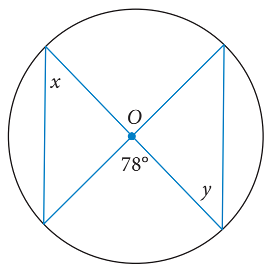
## Concepts and techniques

1 Example 1 Find the value of the pronumerals (*O* is the centre of each circle) in the circles below.

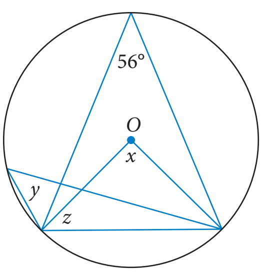
a


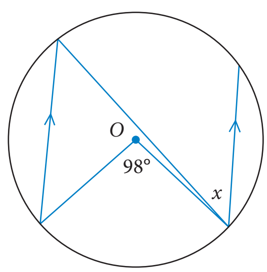

b


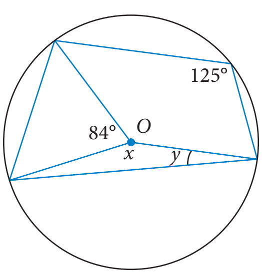

c


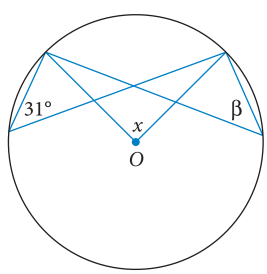

d



2 Find the values of all pronumerals (*O* is the centre of each circle) in the following circles.

a


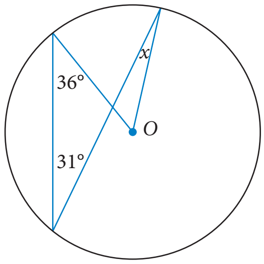

b



c


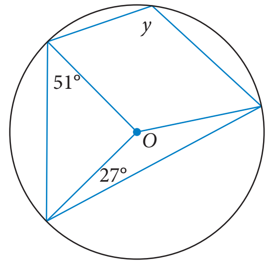

d


3 Find the values of all pronumerals ($O$ is the centre of each circle) in the following circles.

a

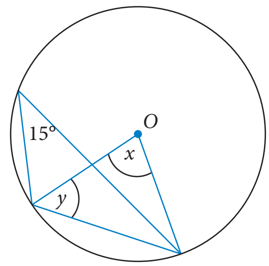



b

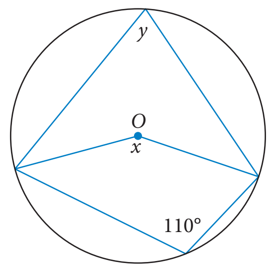



## Reasoning and communication

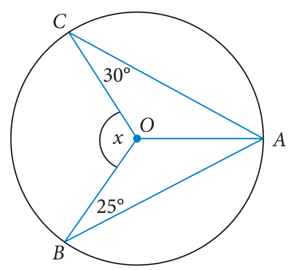
4 Find the value of $x$ and $y$, giving reasons.



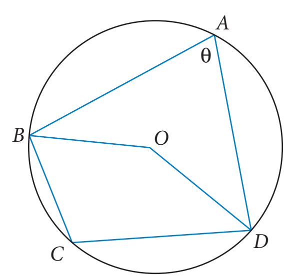

5 Find the value of $x$ and $y$, giving reasons.




6 Find the value of $x$. Give reasons for each step in your calculation.




7 The circle has centre $O$ and $\angle DAB = \theta$.
Show that $\angle DAB$ and $\angle BCD$ are supplementary.





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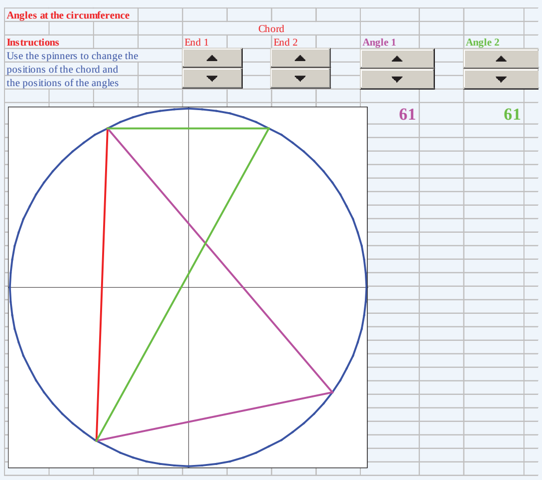
# 6.02 ANGLES AT THE CIRCUMFERENCE OF CIRCLES

## TECHNOLOGY Angles at the circumference

Angles at the circumference

The spreadsheet 'Angles at the circumference' is available on NelsonNet. You can use the spinners to change the positions of the chords and angles. Use the spreadsheet to investigate their relationship.




Consider the situation in the diagram on the right.

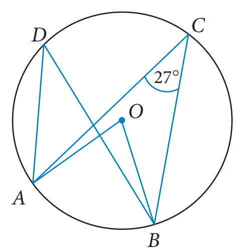
What is $\angle AOB$?

What does that make $\angle ADB$?

What is the relationship between $\angle AOB$ and $\angle ADB$?

Is the relationship still true if you erase the centre and the lines $AO$ and $BO$?

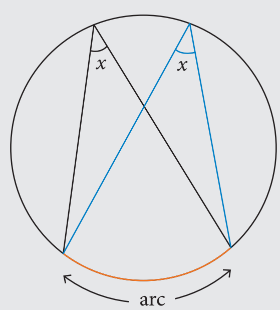
Would this relationship be true for other angles at the circumference standing on the same arc?




**IMPORTANT**

### Theorem 2: Angles on the same arc

**Angles** at the **circumference** of a circle subtended by the **same arc** are **equal**.

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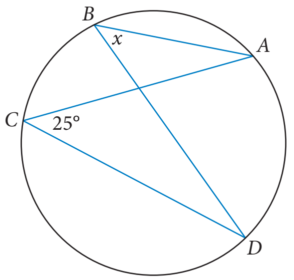
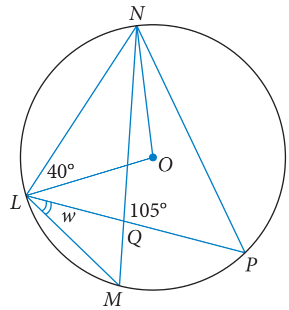
## Example 2

Find the value of the unknowns in the circles below.

a 


b 


### Solution

a $\angle DCA$ and $\angle DBA$ are both standing on arc $AD$. $x = 25°$ (Same arc)

b $ON = OL$ are both radii. $\angle LNO = 40°$ (Isosceles △)

$\therefore \angle LON = 100°$ (△$LNO$ angle sum)

$\angle LPN$ and $\angle LON$ are both on arc $LN$. $\therefore \angle LPN = 50°$ (Angle at centre)

Use △$QPN$ to find $\angle MNP$. $\therefore \angle MNP = 25°$ (△$QPN$ angle sum)

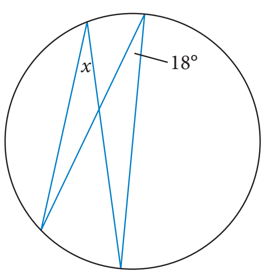
$\angle MNP$ and $w$ are both on arc $PM$. $\therefore w = 25°$ (Angle at centre)

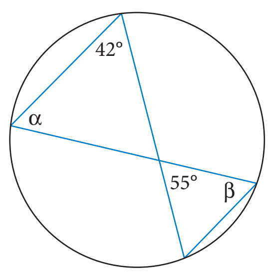
There is no unique solution to examples like the ones shown above. You should aim for the most efficient solution. The solution requiring the least number of steps is the most 'elegant'.

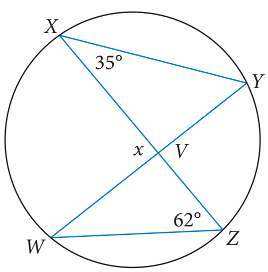
# EXERCISE 6.02 Angles at the circumference of circles

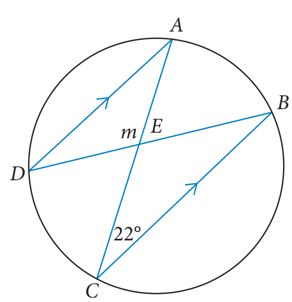
## Concepts and techniques

1 Example 2 Find the values of all pronumerals ($O$ is the centre of each circle) in the circles below.

a 


b 


c 


d 

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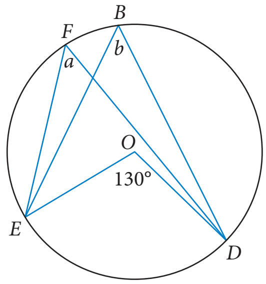
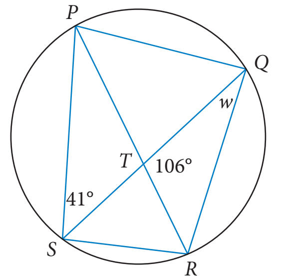
**2** Find the value of the pronumerals in the circles below, giving reasons.

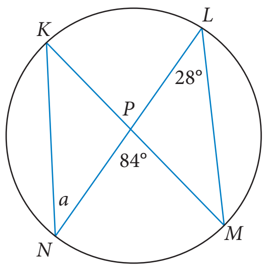
**a**



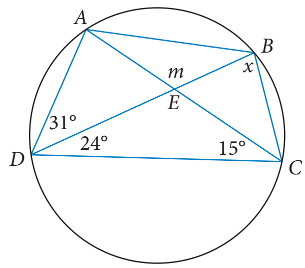

**b**




**c**



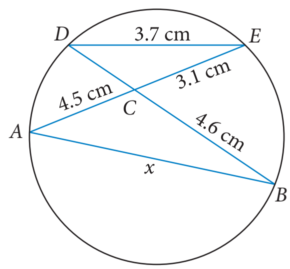

**d**



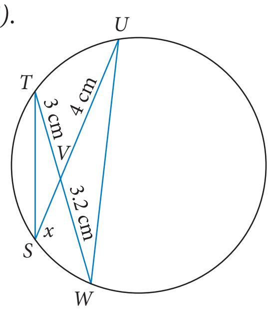

## Reasoning and communication

**3** **a** Prove that $\triangle ABC \,|||\, \triangle DEC$.

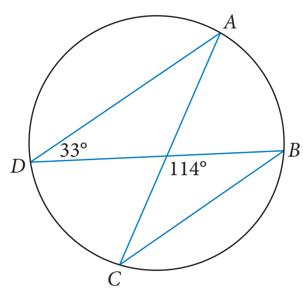
**b** Hence find the value of $x$, correct to 1 decimal place.




**4** Prove that $\triangle STV$ and $\triangle WUV$ are similar. Hence find the length $x$ $(VS)$.




**5** Show that $AD \parallel BC$ in the circle below.



6 (Proof of Theorem 2)

a Construct a circle with arc $AB$ and construct $\angle ACB$ and $\angle ADB$ on the circumference of the circle such that $C$ and $D$ are not on the arc $AB$.

b Mark the centre and draw in the lines $AO$ and $BO$.

c What can you say about $\angle AOB$ and $\angle ACB$?

d What can you say about $\angle AOB$ and $\angle ADB$?

e What does this prove about $\angle ACB$ and $\angle ADB$?

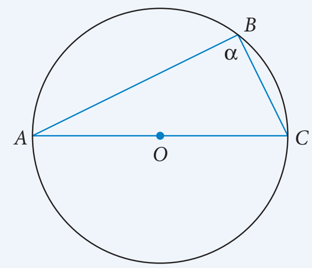
f Set out a formal proof that angles subtended by the same arc on the circumference of a circle are equal.

# 6.03 SEMICIRCLE ANGLE

The angle in a semicircle is a special case of Theorem 1, where the angle at the centre is 180°.

## INVESTIGATION The angle in a semicircle

Consider the circle with diameter $AC$ and point $B$ on the circumference as shown.
Let $\angle ABC = \alpha$.




We know that $AOC$ is a straight line.

What is the size of $\angle AOC$?

Now using Theorem 1, what can you say about $\angle AOC$ and $\angle ABC$?

Therefore what is the size of $\angle ABC$?

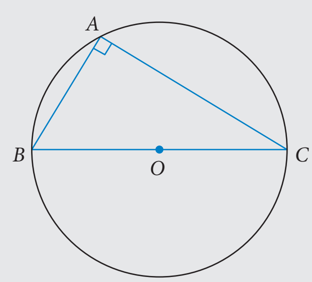
What does this prove about $\alpha$, the angle at the circumference subtended by the semicircle?

**IMPORTANT**

### Theorem 3: Semicircle angle

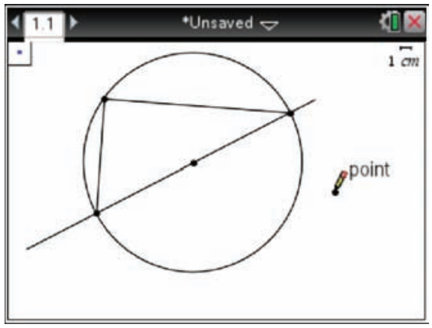
An **angle at the circumference subtended by a diameter** is a **right angle**.

Conversely, a chord that subtends a right angle at the circumference is a diameter.

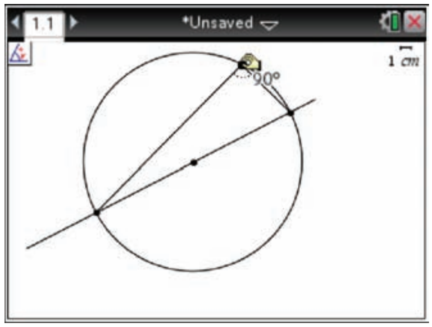


## TI-Nspire CAS

Use a Geometry page.
Use menu, 5: Shapes and 1: Circle to draw a circle.
Use menu, 4: Points & Lines and 4: Line to draw a line from the circumference through the centre.
Use menu, 4: Points & Lines and 3: Intersection Point to select the line and a point on the circumference to extend to the other side of the circle.
Use Line Segment to draw the other lines shown on the screenshot.

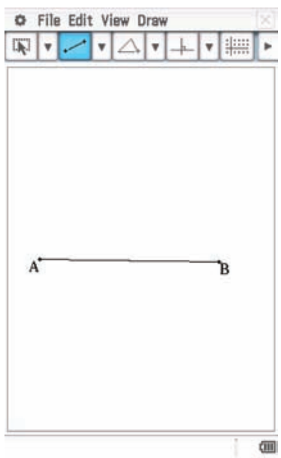



Use 6: Measurement to measure the angle subtended by the diameter (see page 185).
Now grab the point and move it around the circumference. What do you observe?



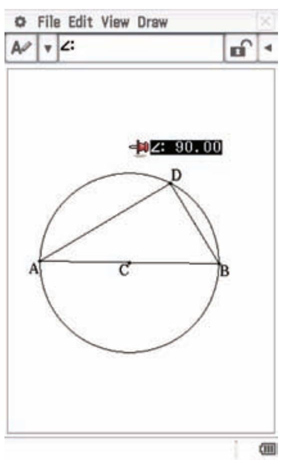

## ClassPad

Use the **Geometry** application.
Use the line segment tool then tap two points on the screen. This will draw a line between points A and B that will become the diameter of a circle.
Tap the selection arrow and select *AB*.

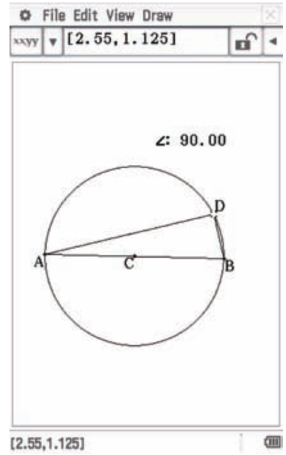



Tap **Draw**, then **Construct**, then **Midpoint**. Point C will appear.

Select the circle construction tool, tap *C* to select it as the centre and then tap either *A* or *B*.
Use the line segment tool to draw *AD* and *BD* so *D* is on the circumference.
Tap the selection arrow and select ∩ *ADB* by tapping *AD* and *BD*.
Tap the top right arrow.
Drag the measurement of ∠ *ADB* down onto the screen.



Tap a blank part of the screen and then tap *D*. Hold the stylus point on *D* then move the point by dragging it. Try moving the other points.





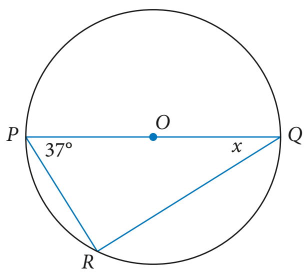


Angles in semicircles and subtended by equal chords

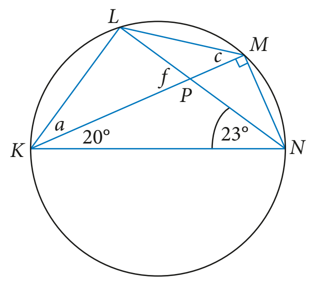
## Example 3

Find the value of the unknowns in the circles below.

**a**




**b**




### Solution

**a** *PQ* is a diameter. $\angle PRQ = 90°$ (Semicircle angle)

Use the angles in the triangle. $x = 53°$ ($\triangle PRQ$ angle sum)

**b** Use the converse of the theorem. *KN* is a diameter (Semicircle angle converse)

Now use the theorem normally. $\therefore \angle KLN = 90°$ (Semicircle angle)

$\therefore a + 20° + 23° + 90° = 180°$ ($\triangle KLN$ angle sum)

$\therefore a = 47°$

23° and *c* are both on the arc *KL*. $c = 23°$ (Same arc)

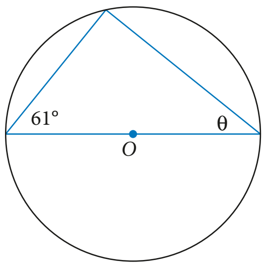
Use the exterior angle rule for *f*. $\therefore f = 43°$ (Exterior angle of $\triangle PKN$)

# EXERCISE 6.03 Semicircle angle

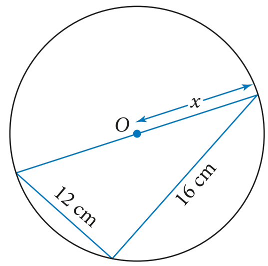
## Concepts and techniques

1 Example 3 Find the values of all pronumerals ($O$ is the centre of each circle) for the following circles.

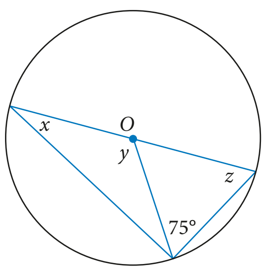
**a**



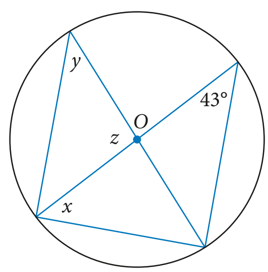

**b**




**c**

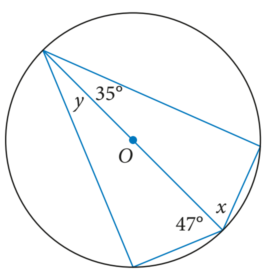



**d**

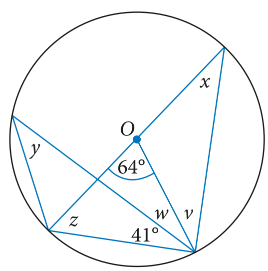



2 Find the values of all pronumerals ($O$ is the centre of each circle) for the following circles.

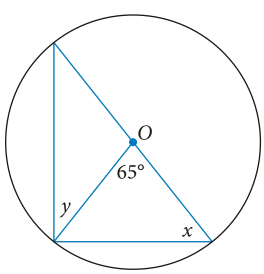
**a**



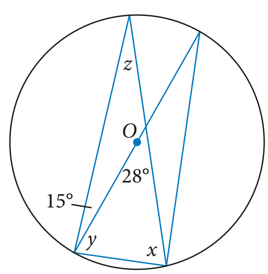

**b**



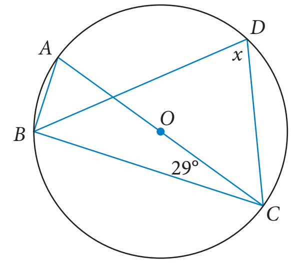

**c**




**d**



3 Evaluate $x$, giving reasons for each step in your working out.




## Reasoning and communication

4 (Proof of the converse of Theorem 3, i.e., that the vertices of a right-angled triangle lie on a circle with the hypotenuse as the diameter.)

a Construct a diagram showing a right-angled triangle $PRQ$, with the right angle at $R$ and the midpoint of $PQ$ at $S$.
Draw in the point $X$ such that $PRQX$ is a rectangle.

b Connect $S$ and $X$.

c Given that $PRQX$ is a rectangle, what can you say about the lengths of $PQ$ and $RX$?

d What is the significance of the point $S$ in relation to $PQ$ and $RX$?

e Therefore, what can you say about the lengths $SR$, $SQ$, $SP$ and $SX$?

f Therefore, $S$ is the centre of the circle passing through which points?

g Set out a formal proof that the vertices of a right-angled triangle lie on a circle with the hypotenuse as the diameter.

5 Alternate proof of an angle in a semicircle.

a Construct a diagram showing a circle with $O$ as the centre, $AB$ as the diameter and a point $C$ on the circumference such at $ABC$ forms a triangle. Let $\angle ACO = \alpha$ and $\angle BCO = \beta$.

b What sort of triangles are $ACO$ and $BCO$?

c Which angle is equal to $\angle ACO$?

d Which angle is equal to $\angle BCO$?

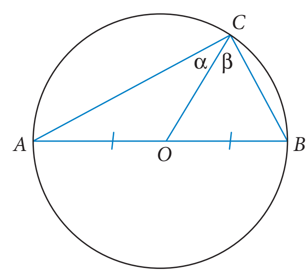
e What is the size of $\angle AOC$ in terms of $\alpha$?

f What is the size of $\angle BOC$ in terms of $\beta$?

g What is the size of $\angle AOB$ in terms of $\alpha$ and $\beta$?

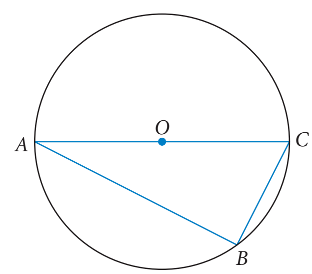
h Complete the statement: $2\alpha + 2\beta =$ ____________°
$\therefore\ \alpha + \beta =$ ____________°

i Set out a formal proof that the angle in a semicircle is 90°.




6 $AB = 6$ cm and $BC = 3$ cm. $O$ is the centre of the circle.

Show that the radius of the circle is $\frac{3\sqrt{5}}{2}$ cm.





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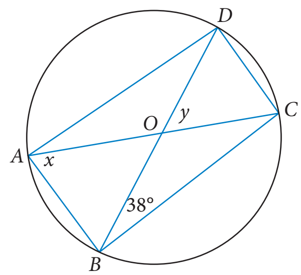
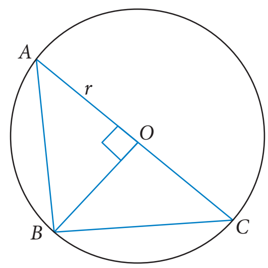
7 The circle has centre $O$.
a Evaluate $x$ and $y$.
b Show that $AD = BC$.




8 A circle has centre $O$ and radius $r$ as shown.
a Show that triangles $AOB$ and $ABC$ are similar.
b Show that $BC = \sqrt{2}r$.




# 6.04 ARCS AND CHORDS

It is possible to relate equal chords or equal arcs to the angle subtended at the centre of a circle.

## INVESTIGATION Equal chords subtending equal angles

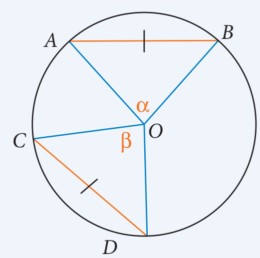
Consider a circle, centre $O$, with chords $AB$ and $CD$ as shown.
Let $\angle AOB = \alpha$ and $\angle COD = \beta$.

Which congruence test would you use to prove that $\triangle AOB$ and $\triangle COD$ are congruent?

What can you say about $\angle AOB$ and $\angle COD$?

What does this prove about the angles subtended at the centre of a circle by equal chords?

Is the converse true?

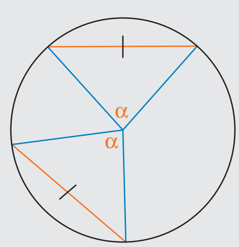



**IMPORTANT**

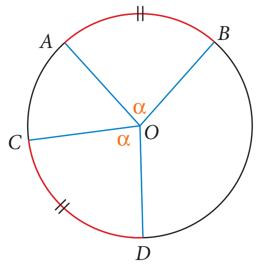
**Theorem 4: Centre angles on equal chords**

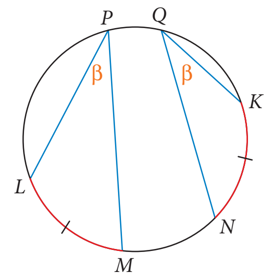
Equal chords of a circle subtend equal angles at the centre.

Conversely, equal angles at the centre subtend equal chords.



It follows that the angles subtended at the centre or at the circumference by equal arcs are also equal. Conversely, equal angles at the centre subtend equal arcs.

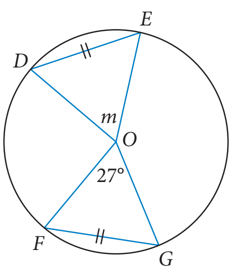


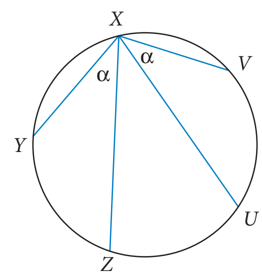




## Example 4

**a** Find the value of $m$.

**b** In the diagram, $YZ = 15$ cm. Find the length of $UV$.




### Solution

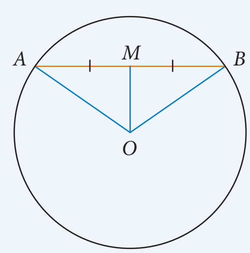
**a** $DE = FG$ $\quad$ $m = 27°$ (Equal chords)

**b** $\angle YXZ = \angle UXV$ $\quad$ $UV = 15$ cm (Equal chords/arcs converse)

Another theorem relates the perpendicular bisector of a chord to the centre of the circle.

## INVESTIGATION The bisector of a chord passing through the centre of a circle

Consider a circle, centre $O$, with chord $AB$. Let $M$ be the midpoint of $AB$.




Join $AO$, $BO$ and $OM$.

Which congruence test would you use to prove that $\triangle AMO$ and $\triangle BMO$ are congruent?

Why are $\angle AMO$ and $\angle BMO$ equal?

What is the sum of $\angle AMO$ and $\angle BMO$?

Therefore, what is the size of $\angle AMO$?

What does this say about the angle between the bisector of a chord and the chord?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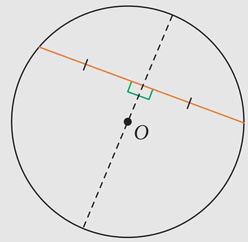
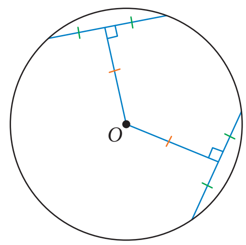
IMPORTANT

**Theorem 5: Chord bisector**

A radius that bisects a chord is perpendicular to the chord.




The following results (corollaries) follow from the previous theorem.




**1** The perpendicular bisector of a chord passes through the centre of a circle.

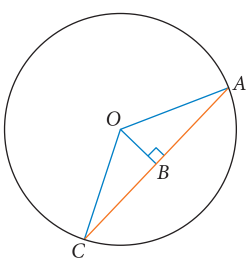
**2** The line passing through the centre of a circle perpendicular to a chord bisects the chord.

**3** The perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.

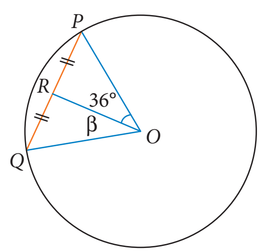
**4** Equal chords are equidistant from the centre of a circle.

## Example 5

**a** $AC = 16$ cm and $OB = 5$. Find the radius.




**b** Find the value of $\beta$.




### Solution

**a** $OB \perp AC$, so $B$ is the midpoint of $AC$. — $AB = 8$ cm (Chord bisector corollary)

Use Pythagoras' theorem. — $OA^2 = OB^2 + AB^2$

Substitute values. — $= 5^2 + 8^2 = 89$

Write the answer. — Radius $= \sqrt{89}$

**b** $PR = QR$, so $OR$ is the perpendicular bisector of $PQ$. — $\beta = 36°$ (Chord bisector corollary)

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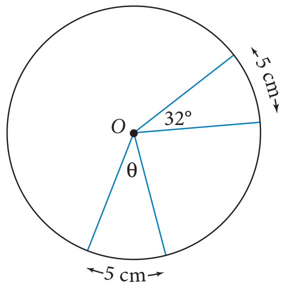
# EXERCISE 6.04 Arcs and chords

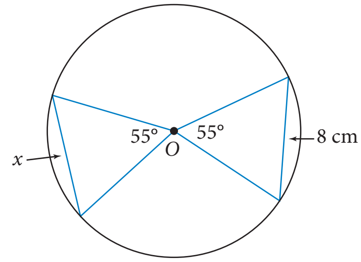
## Concepts and techniques

1 Example 4 Find the values of all pronumerals ($O$ is the centre of each circle) in the following circles.

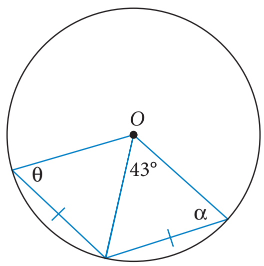
a



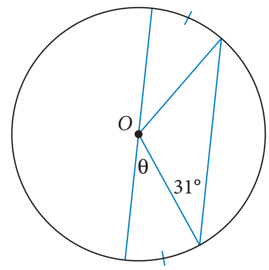

b



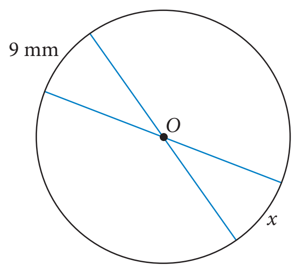

c



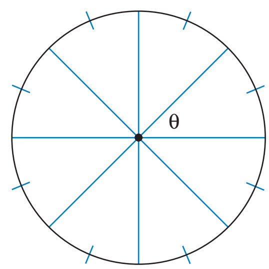

d




e

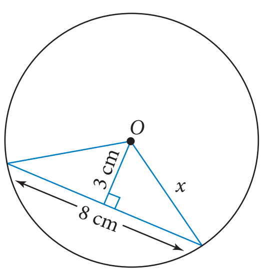



f

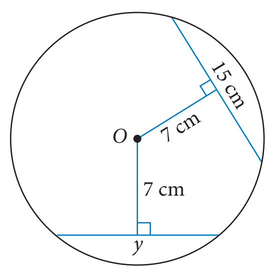



2 Example 5 Find the values of all pronumerals ($O$ is the centre of each circle) in the following circles.

a




b



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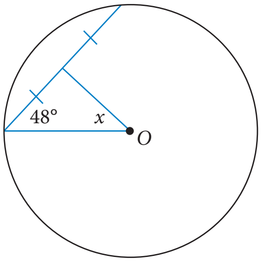
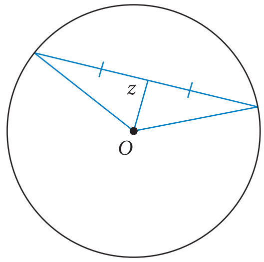
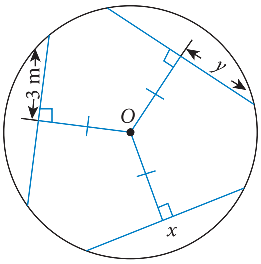
**c**



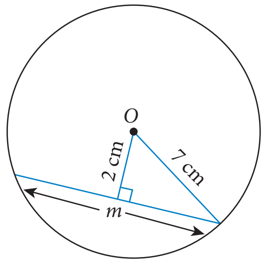

**d**




**e**




**f**



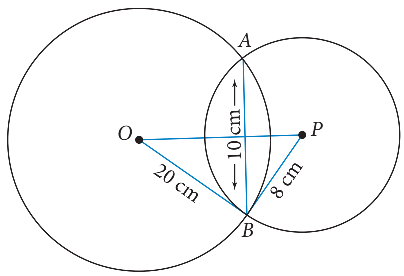

**3** Find the exact radius of a circle with a chord that is 8 cm long and 5 cm from the centre.

**4** A circle with radius 89 mm has a chord drawn 52 mm from the centre. How long, to the nearest millimetre, is the chord?

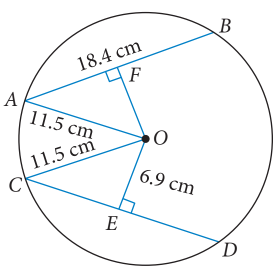
## Reasoning and communication

**5** In this circle, *O* and *P* are the centres of intersecting circles with radii 20 cm and 8 cm respectively. *AB* is the common chord through the intersections of the circles.

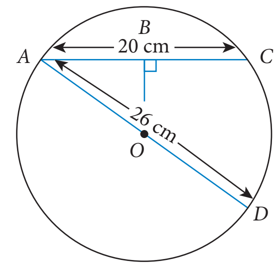
Prove that $\triangle OAP \equiv \triangle OBP$ and hence find the distance *OP* correct to 1 decimal place.




**6** In the circle below, show that $AB = CD$.



7 In the circle on the right, $AC = 20$ cm and $AD = 26$ cm. Find $OB$, correct to 1 decimal place.




8 [Proof that the perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.]

a Construct a chord $AB$ on a circle with centre $O$. Join $AO$ and $BO$. Join the midpoint $M$ of $AB$ to $O$.

b Use a congruence test to prove that $\triangle AMO \equiv \triangle BMO$.

c What can you say about $\angle AOM$ and $\angle BOM$?

d Set out a proof that shows that the perpendicular bisector of a chord bisects the angle at the centre of the circle subtended by the chord.

9 [Proof that equal chords are equidistant from the centre of a circle.]

a Construct two equal non-intersecting chords in a circle. Label them $AB$ and $CD$. The centre of the circle is $O$. Construct $AO$ and $CO$.

b Join the midpoint $F$ of $AB$ to the centre $O$. Join the midpoint $G$ of $CD$ to the centre $O$.

c Use a congruence test to prove that $\triangle AFO \equiv \triangle CGO$.

d What can you say about the lengths $FO$ and $GO$?

e Set out a formal proof that equal chords are equidistant from the centre of a circle.

# 6.05 INTERSECTING CHORDS

If two chords intersect, we can relate the lengths of the resulting intercepts.

## INVESTIGATION The intervals of intersecting chords

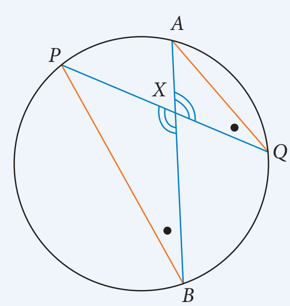
Consider the circle with chords $AB$ and $PQ$ intersecting at $X$ as shown. Join $PB$ and $AQ$.

Which similarity test would you use to prove that the resulting triangles $\triangle PBX$ and $\triangle AQX$ are similar?

What can you say about the ratios of the matching sides $\frac{PB}{AQ}, \frac{BX}{QX}$ and $\frac{PX}{AX}$?

Why does $AX \cdot BX = PX \cdot QX$?

What does this say about the products of the lengths of the intervals on two intersecting chords?





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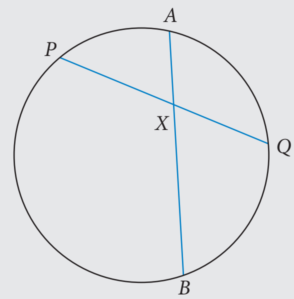
**IMPORTANT**

### Theorem 6: Intersecting chords

When two chords of a circle intersect, the **product of the lengths** of the intervals on one chord **equals** the **product of the lengths** of the intervals on the other chord.

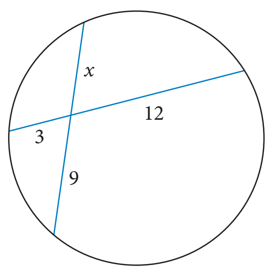
In the circle, $AX \cdot BX = PX \cdot QX$



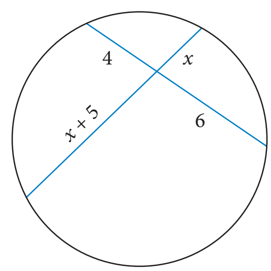

## Example 6

Find the value of the unknowns in the following circles.

**a**




**b**




### Solution

**a** Use the theorem. $x \times 9 = 3 \times 12$ (Intersecting chords)

Solve the equation. $9x = 36$

$x = 4$

**b** Use the theorem. $x(x + 5) = 4 \times 6$ (Intersecting chords)

Expand brackets. $x^2 + 5x = 24$

Factorise. $x^2 + 5x - 24 = 0$

Find the values of $x$. $(x - 3)(x + 8) = 0$

$x = 3$ or $x = -8$

A length cannot be negative. So $x \neq -8$

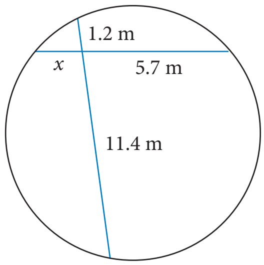
Write the answer. $x = 3$.

# EXERCISE 6.05 Intersecting chords

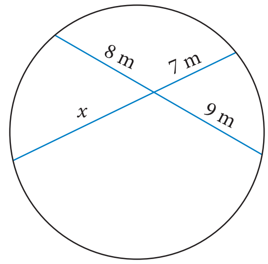
## Concepts and techniques

1 Example 6 Find the values of all pronumerals ($O$ is the centre of each circle) in the following circles.

a




b

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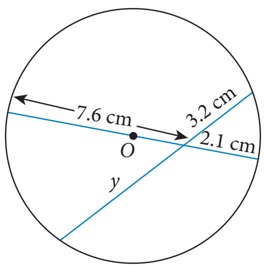


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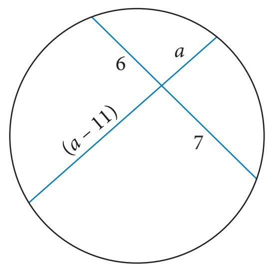
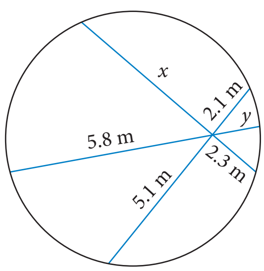



2 Find unknowns in the circles below correct to 1 decimal place if necessary.

a

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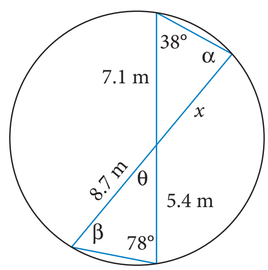
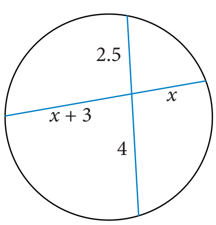



3 Find the pronumerals in the circles below.

a

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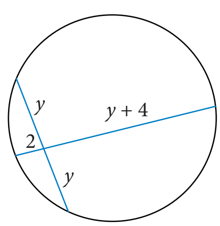
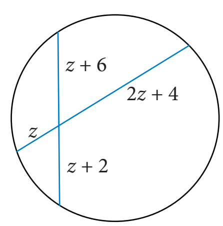



c





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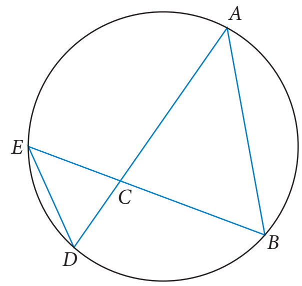
## Reasoning and communication

4 Theorem proof

a Prove that triangles $ABC$ and $EDC$ are similar.

b Show that $AC \cdot CD = BC \cdot CE$.

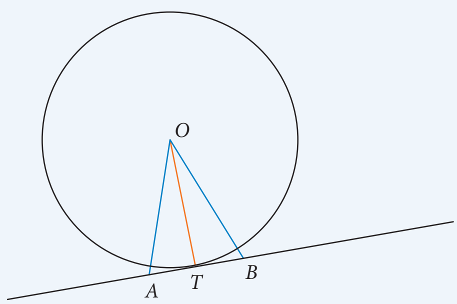



# 6.06 TANGENTS AND SECANTS

There are a number of theorems relating a tangent to a circle.

## INVESTIGATION The relationship between the radius and a tangent at the point of contact

Consider a circle where $T$ is the point of contact of a tangent with the circle. $A$ and $B$ are any other points on the tangent as shown. Join line segments $AO$, $BO$ and $TO$.




$TO$ is a radius because $T$ lies on the circle.

Why are $AO$ and $BO$ longer than the radius $TO$?

By definition, what is the shortest distance from a point to a line?

What is the shortest distance from $O$ to the tangent?

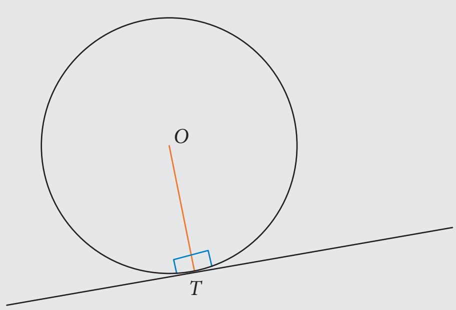
Why is $TO$ perpendicular to the tangent?

What does this say about the angle between a tangent and a radius at the point of contact?

**IMPORTANT**

### Theorem 7: Tangent and radius

A **tangent** drawn to a circle is **perpendicular** to the **radius** at the point of contact.

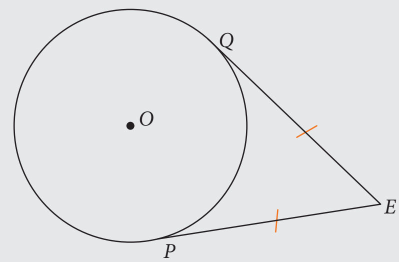


The next theorem relates the two tangents that can be drawn to a circle from an external point. You will prove it in Exercise **6.06**.

**IMPORTANT**

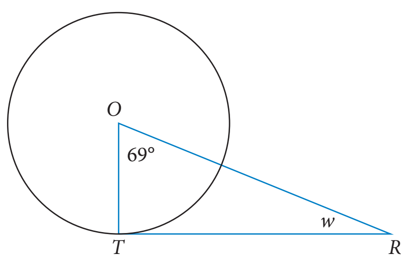
### Theorem 8: External tangents

**Tangents** drawn to a circle from an **external** point are **equal** in length.



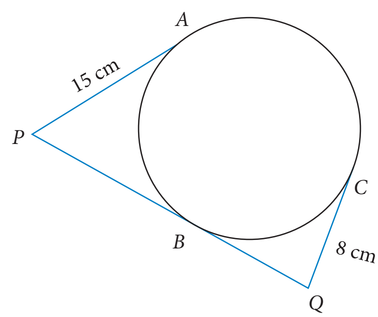

## Example 7

**a** $TR$ is a tangent.




Find the value of $w$.

**b** The points $A$, $B$ and $C$ lie on the circle.




Find the length of $PQ$.

### Solution

**a** Find a second angle.

$\angle OTR = 90°$ (Tangent and radius)

$w + 69° = 90°$ ($\triangle OTR$ angle sum)

$w = 21°$

**b** Use the tangents.

$PB = 15$ cm (External tangents)

Use the other tangents.

$QB = 8$ cm (External tangents)

$\therefore PQ = 15 + 8$

$= 23$ cm

The next theorem is known as *the alternate segment* theorem.

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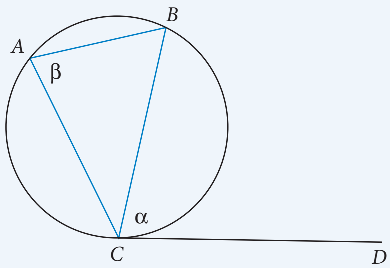
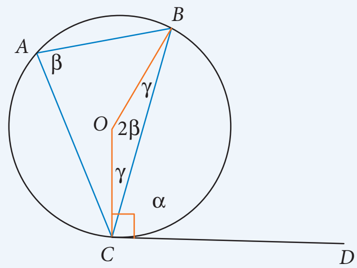
## INVESTIGATION The alternate segment theorem

Consider the circle with chord *CB* and tangent *CD* meeting at *C* as shown.




Let $\angle BCD = \alpha$, $\angle BAC = \beta$. Join *OC* and *OB*. Let $\angle OCB = \gamma$.

What is the relationship between $\angle COB$ and $\angle BAC$?

What type of triangle is *COB*?

Explain why $\angle OBC = \gamma$.

What can you say about $\angle OCD$?

Explain why $\beta + \gamma = 90°$ .

Explain why $\alpha + \gamma = 90°$

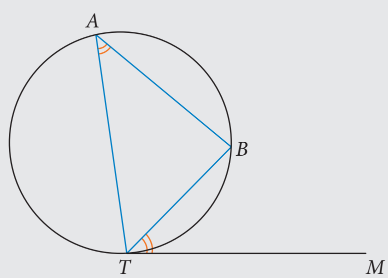
What does this say about $\alpha$ and $\beta$?

**IMPORTANT**

### Theorem 9: Alternate segment

The angle between the tangent and a chord is equal to the angle in the alternate segment.

In the diagram, $\angle TAB = \angle BTM$.




A further theorem relates a tangent to a secant. You will prove this in Exercise **6.06**.

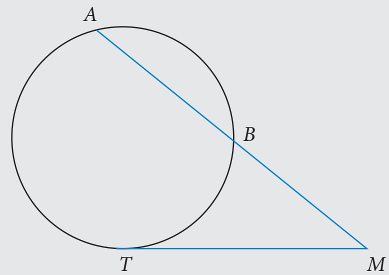
Intersecting chords, radii, tangents and secants

**IMPORTANT**

### Theorem 10: Tangent and secant

When a **secant** and a **tangent** are drawn to a circle from an **external** point, the **square** of the length of the **tangent** equals the **product** of the **lengths** to the circle on the **secant**.

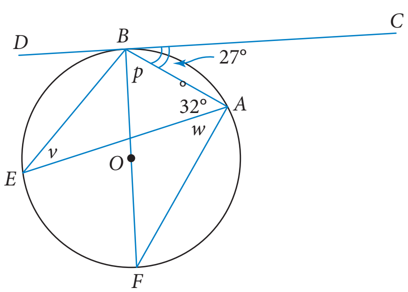
In the diagram, $AM \cdot BM = TM^2$



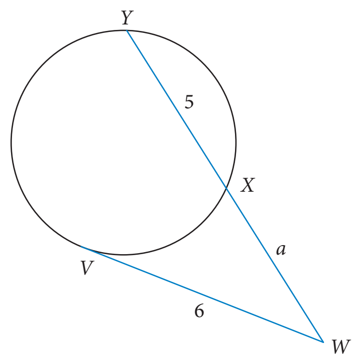

## Example 8

Find the values of the unknowns.

**a** $DC$ is a tangent.




**b** $VW$ is a tangent.




### Solution

**a** $BF$ is a diameter. — $\angle BAF = 90°$ (Semicircle angle)

Use the right angle. — $w = 58°$ (Complementary)

$BO$ is a radius. — $\angle CBO = 90°$ (Tangent and radius)

$p = 63°$ (Complementary)

Look at $BC$ and $\triangle BEA$. — $v = 27°$ (Alternate segment)

**b** Use the theorem. — $a(a + 5) = 6^2$ (Tangent and secant)

Expand brackets. — $a^2 + 5a = 36$

Write in standard form. — $a^2 + 5a - 36 = 0$

Factorise. — $(a - 4)(a + 9) = 0$

$a = 4$ or $a = -9$

$a$ is a length. — But $a \geq 0$, since $a$ is a length, so $a \neq -9$.

Write the answer. — $a = 4$.



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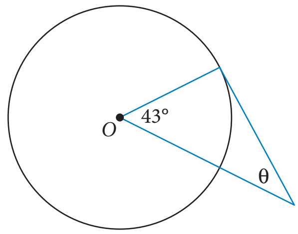
# EXERCISE 6.06 Tangents and secants

## Concepts and techniques

1 Example 7 Find the values of the pronumerals in the following circles.

a
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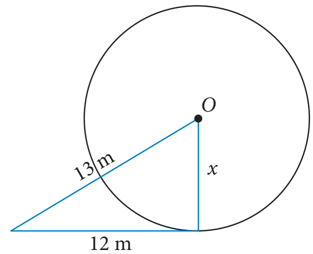
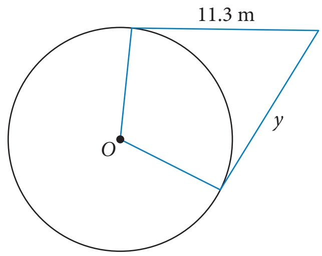



c
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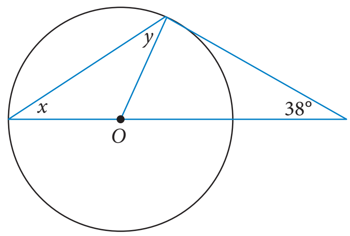
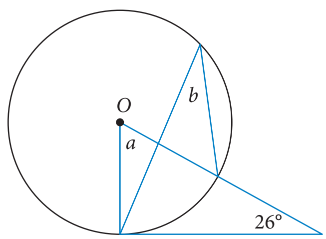
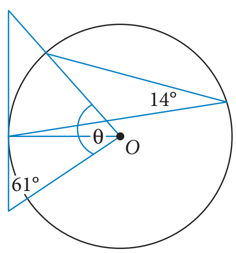
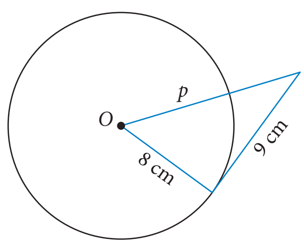



e



f



g
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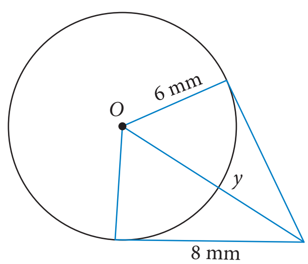


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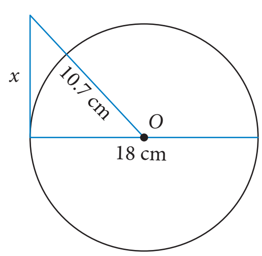


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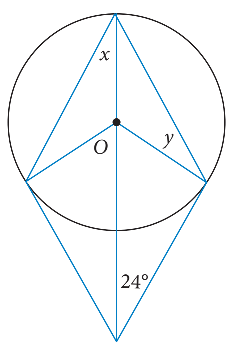
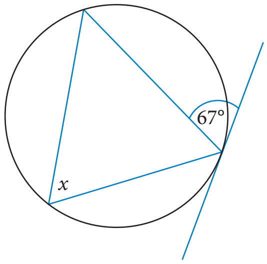


2 Example 8 Find the values of all pronumerals ($O$ is the centre of each circle; all external lines are tangents) in the following circles.

a

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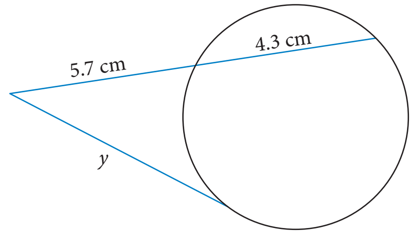
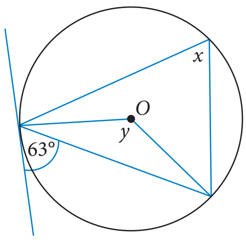



c

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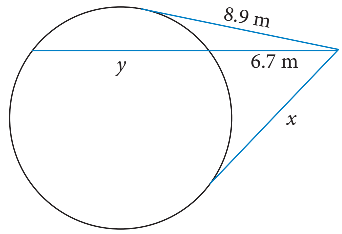
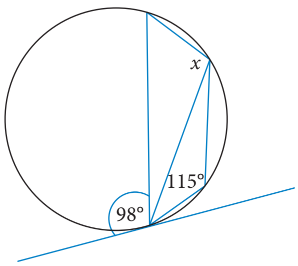



e

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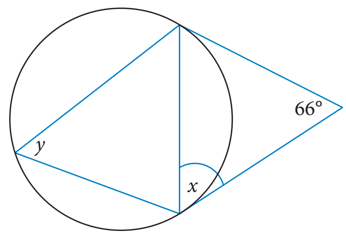


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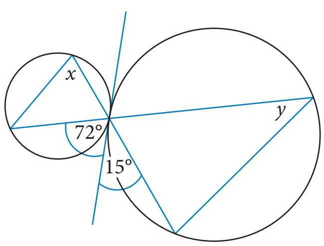


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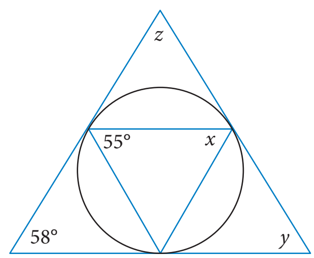
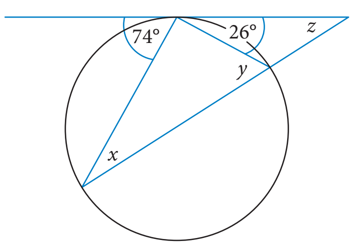



3 Find the values of all pronumerals in the following circles.

a

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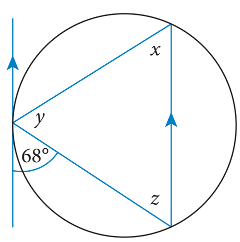




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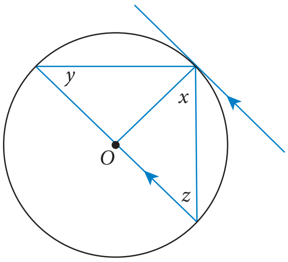


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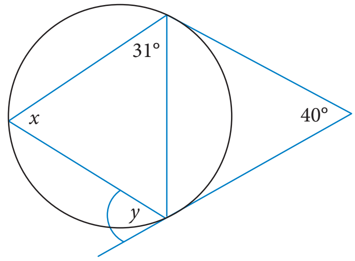


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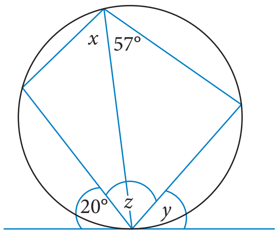


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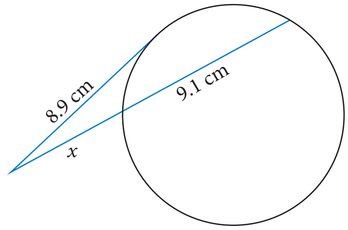


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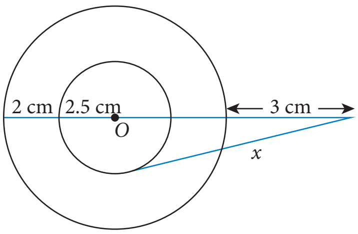


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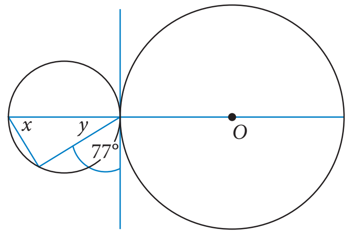


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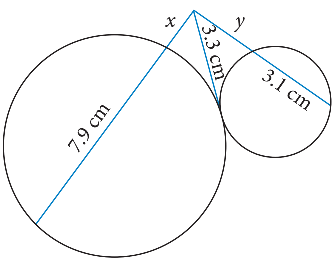
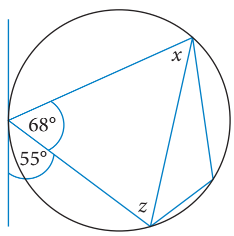
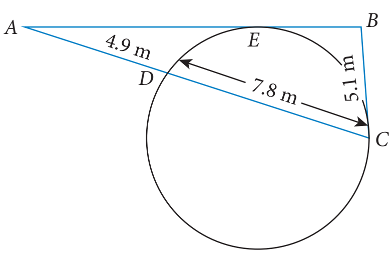



j




## Reasoning and communication

4 Find $AB$, given $AD = 4.9$ m, $BC = 5.1$ m and $CD = 7.8$ m.



5 [Proof of Theorem 8]

a Construct a circle with tangents $EQ$ and $EP$, where $E$ is an external point and $Q$ and $P$ are the points of contact of the tangents with the circle.

b Join $OP$, $OQ$ and $OE$.

c Use a congruence test to prove that $\triangle POE$ and $\triangle QOE$ are congruent.

d Explain why $PE$ and $QE$ are equal in length.

e Set out a proof that shows that tangents drawn to a circle from an external point are equal in length.

6 [Proof of Theorem 10]

a Construct a circle with tangent $TM$ meeting the circle at $T$. Mark a point $A$ on the circumference and join $A$ to $M$, meeting the circle at $B$.

b Join $AT$ and $TB$.

c Use a similarity proof to show that $\triangle TAM \;|||\; \triangle BTM$.

d Which test did you use?

e Write a statement about the matching sides.

f Explain why $\dfrac{AM}{TM} = \dfrac{TM}{BM}$.

g Is it true that $AM \cdot BM = TM^2$?

h Set out a proof that shows that the square of the length of the tangent to a circle is equal to the product of the lengths to the circle on the secant.

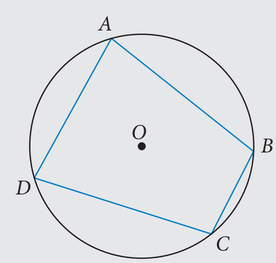
## 6.07 FIGURES IN CIRCLES

You saw in Section **6.03** that the angle in a semicircle is a right angle. This theorem relates to a right-angled triangle inscribed in a circle. There are also theorems related to quadrilaterals inscribed in circles. These are called *cyclic quadrilaterals*.

**IMPORTANT**

A **cyclic quadrilateral** is a quadrilateral whose four vertices lie on the circumference of a circle.

In the diagram, $ABCD$ is a cyclic quadrilateral.



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## INVESTIGATION Opposite angles in a cyclic quadrilateral

Consider the circle with cyclic quadrilateral *ABCD* as shown.
Join *OB* and *OD*.

Let $\angle DAB = \alpha$ and $\angle DCB = \beta$.

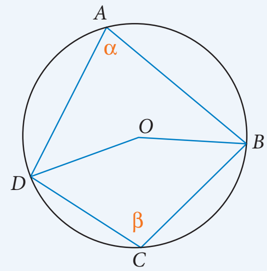
What is the size of obtuse $\angle DOB$ standing on arc *DCB*?

What is the size of reflex $\angle DOB$ standing on arc *DAB*?

Explain why $2\alpha + 2\beta = 360°$.

What does this say about the sum $\alpha + \beta$?

What does this say about the opposite angles in a cyclic quadrilateral?

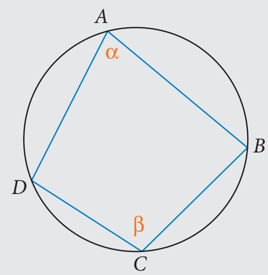



**IMPORTANT**

### Theorem 11: Opposite angles of a cyclic quadrilateral

The **opposite angles** of a cyclic quadrilateral are **supplementary**.

In the diagram, $\alpha + \beta = 180°$.

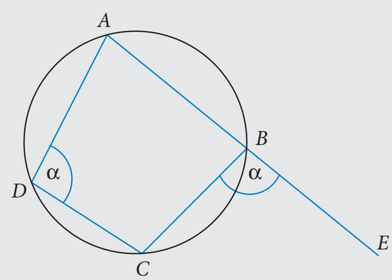



The next theorem will be proved in Exercise **6.07**.

**IMPORTANT**

### Theorem 12: Exterior angle of a cyclic quadrilateral

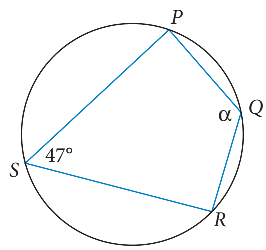
The **exterior angle** of a cyclic quadrilateral is **equal** to the **interior opposite angle**.



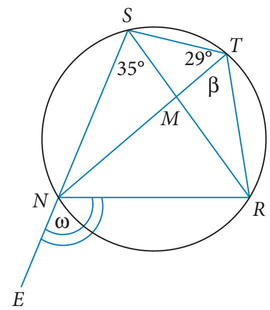

## Example 9

Find the value of the unknown angles in the circles below.

**a**




**b**




### Solution

**a** $PQRS$ is a cyclic quadrilateral. $\alpha = 180° - 47°$ (Opposite angles of cyclic quadrilateral)
$= 133°$

**b** $\beta$ and 35° are both on $NR$. $\beta = 35°$ (Same chord)

Find the opposite angle. $\angle STR = 35° + 29° = 64°$

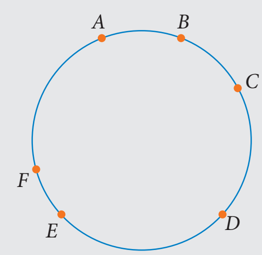
Find the exterior angle. $\omega = 64°$ (Exterior angle of cyclic quadrilateral)

It is possible to draw a circle through any three non-collinear points. Special conditions must apply, however, for four or more non-collinear points to lie on a circle. These conditions relate to the converses of Theorems 11 and 12. The converses of Theorems 11 and 12 are also true.

**IMPORTANT**

Four or more points are called ***concyclic*** if they all lie on the same circle.

In the diagram, $A$, $B$, $C$, $D$, $E$, $F$ are concyclic.

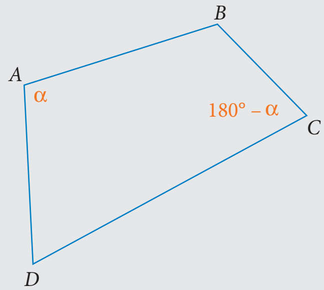


**IMPORTANT**

### Converse of Theorem 11: Opposite angles of a cyclic quadrilateral

If the opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.

In the diagram, *ABCD* is a cyclic quadrilateral.
The four vertices *A*, *B*, *C*, *D* are concyclic.

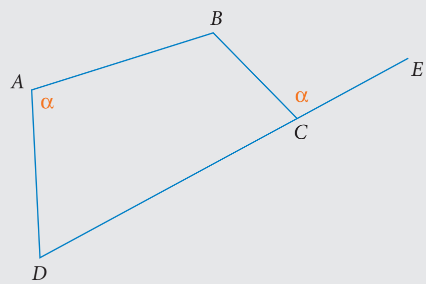



**IMPORTANT**

### Converse of Theorem 12: Exterior angle of a cyclic quadrilateral

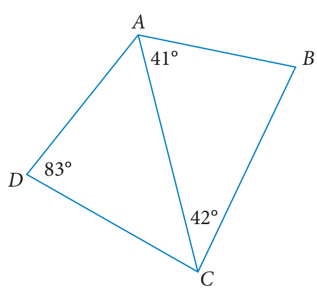
If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.

In the diagram, *ABCD* is a cyclic quadrilateral.




## Example 10

Prove that the points *A*, *B*, *C* and *D* form a cyclic quadrilateral.




### Solution

Find $\angle ABC$.

$\angle ABC = 180° - (41 + 42)°$ ($\triangle ABC$ angle sum)
$= 97°$

Show the opposite angles are supplementary.

Now $\angle ADC + \angle ABC = 83° + 97°$
$= 180°$

Write the conclusion.

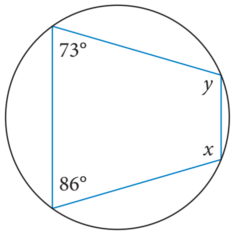
*ABCD* is a cyclic quadrilateral as its opposite angles are supplementary.

# EXERCISE 6.07 Figures in circles

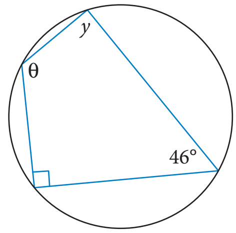
## Concepts and techniques

1 Example 9 Find the values of all pronumerals in the following circles.

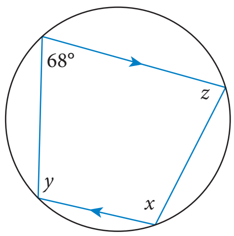
a



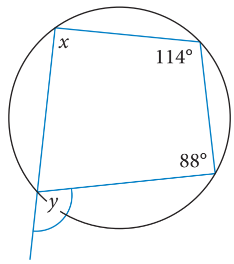

b



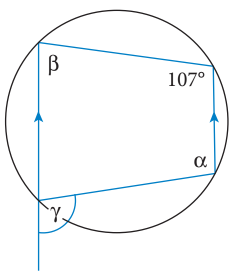

c




d




e

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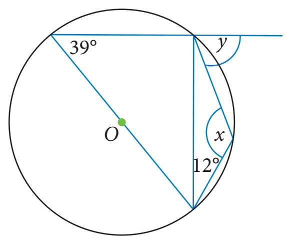


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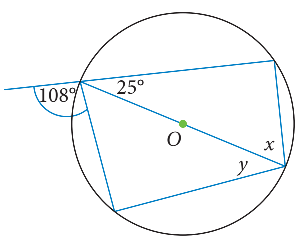


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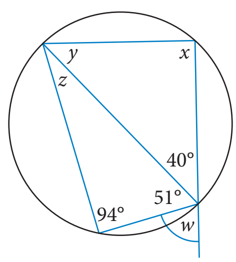


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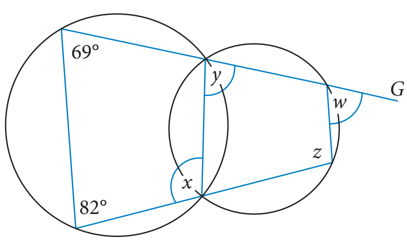


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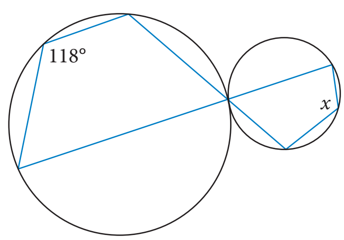




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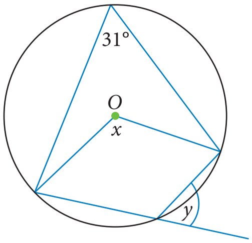
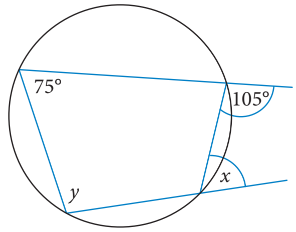
2 Find the values of all pronumerals in the following circles.

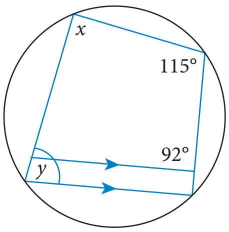
a




b




c

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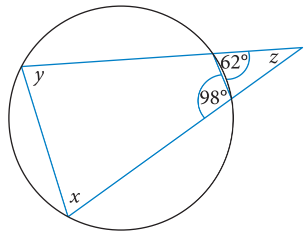


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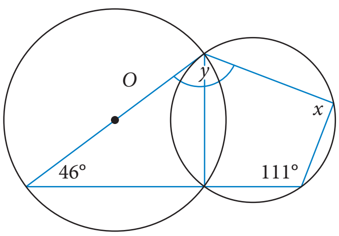


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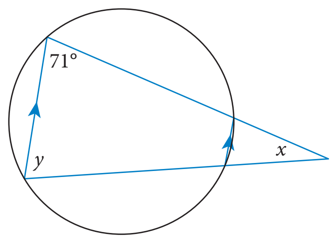


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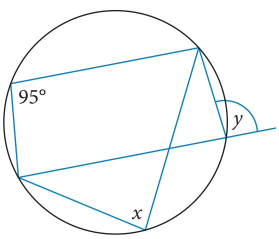


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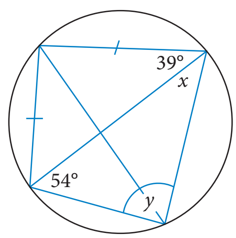


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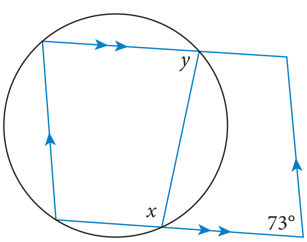


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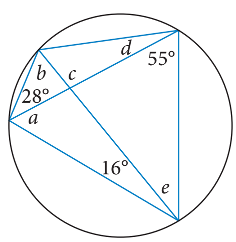
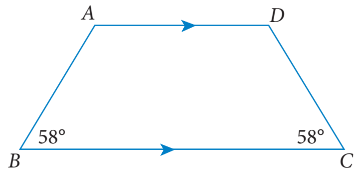



## Reasoning and communication

3 Example 10 Show that *ABCD* is a cyclic quadrilateral.

a

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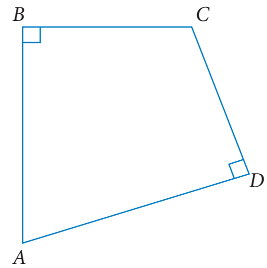


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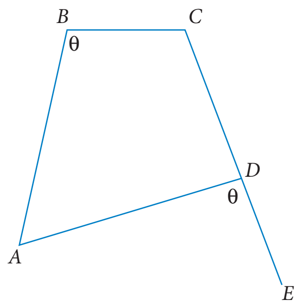


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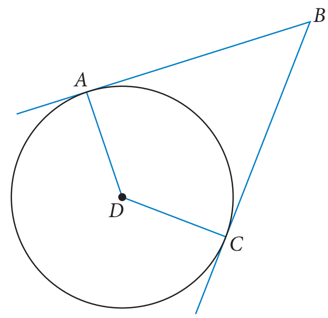



4 [Proof of Theorem 12]

a Construct a circle with cyclic quadrilateral *ABCD*. Produce *AB* to *E*.

b What angle is supplementary with $\angle ADC$?

c What angle is supplementary with $\angle CBE$?

d What can you say about $\angle ADC$ and $\angle CBE$?

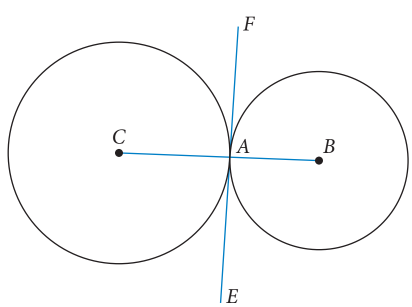
e Set out a formal proof to show that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

# 6.08 MIXED CIRCLE PROBLEMS

You can use several theorems to find unknown angles and lengths.

### Example 11

Prove that the centres and point of contact of two circles that just touch are collinear.




Prove that *C*, *A*, *B* are collinear.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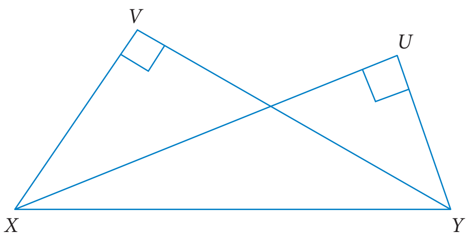
### Solution

| | |
|---|---|
| Find part of $\angle CAB$. | $\angle FAC = 90°$ (Tangent and radius) |
| Find the other part. | $\angle FAB = 90°$ (Tangent and radius) |
| Find the whole angle. | $\angle CAB = \angle FAB + \angle FAC = 180°$ |
| Use $\angle CAB$. | *CAB* is a straight line |
| Write the conclusion. | *C, A, B* are collinear. QED |

## Example 12

Prove that *X, Y, U, V* are concyclic.




### Solution

| | |
|---|---|
| Consider *XUY*. | *XY* is a diameter on a circle through *X, Y* and *U* (Semicircle converse) |
| Consider *XVY*. | *XY* is a diameter on a circle through *X, Y* and *V* (Semicircle converse) |
| Use the common diameter. | The circles must be the same. |
| Write the conclusion. | *X, Y, U, V* are concyclic. QED |

## INVESTIGATION Circle geometry in real-life

National Library of Australia (nla.pic-an7650469)

In 1957, Jorn Utzon won the design competition for Sydney's new opera house which was to be built on Bennelong Point. He designed shells of 'undefined geometry' and it became known as one of the greatest engineering and architectural challenges to resolve the shells into a shape that could be built.

The breakthrough finally came with what is now known as *Utzon's Sphere.*

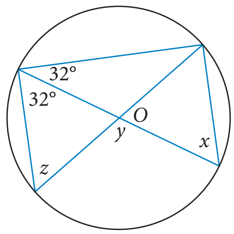
Find out about the design of the Sydney Opera House and the spherical shells that signify it. Can you build it?

# EXERCISE 6.08 Mixed circle problems

## Concepts and techniques

1 Find the values of all pronumerals ($O$ is the centre of each circle) in the circles below.

a

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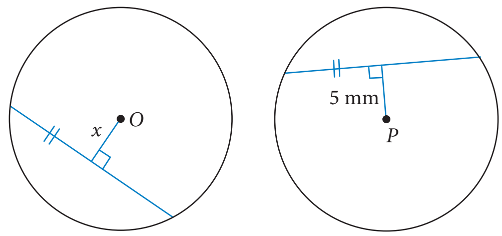
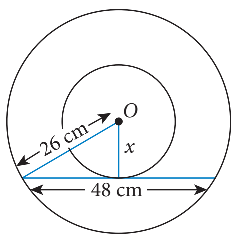
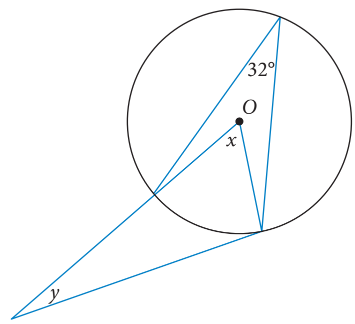



2 Find the values of all pronumerals (all external lines are tangents to the circles) in the following diagrams.

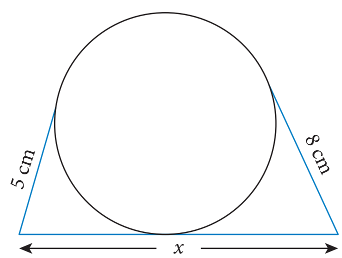
a




b




c

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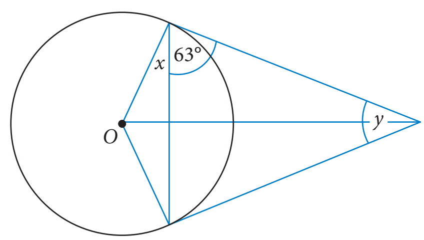
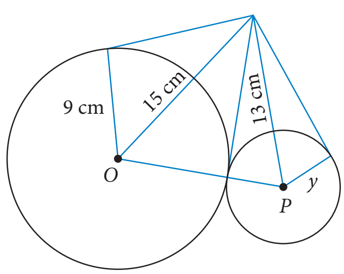



e

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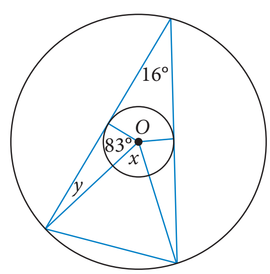




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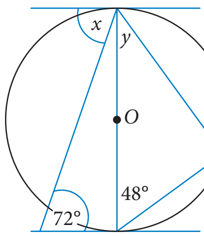
g

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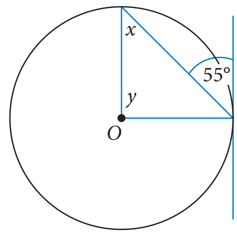
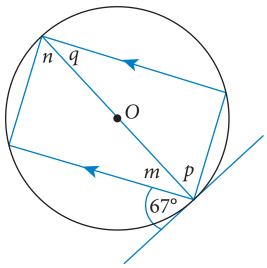



i

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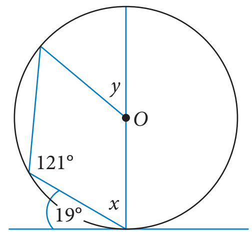
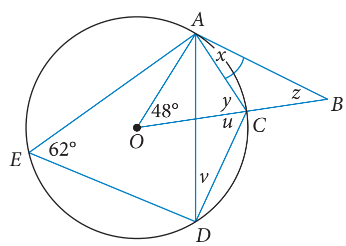
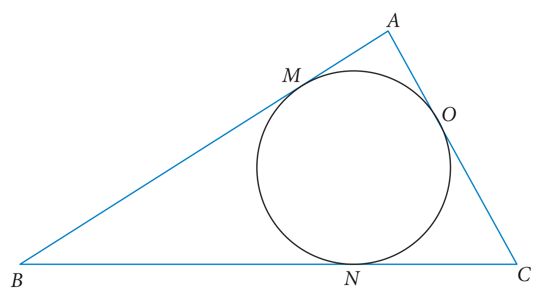


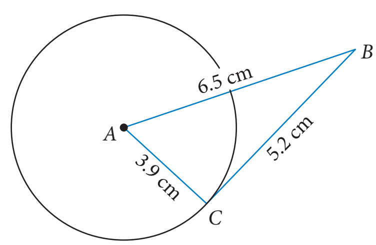

3 Find the values of all pronumerals, giving reasons for each step of your working ($O$ is the centre of the circle, $AB$ is a tangent).




4 $AB$, $BC$ and $AC$ are tangents, with $AB = 24$ cm, $BC = 27$ cm and $BM = 15$ cm. Find the length of $AC$.

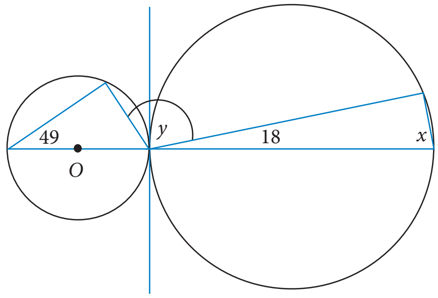



5 Example 11 $AB = 6.5$ m, $AC = 3.9$ m and $BC = 5.2$ m. Prove that $A$ lies on a diameter of the circle, given that $BC$ is a tangent to the circle.

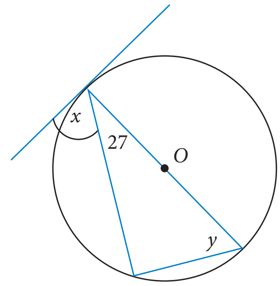



6 Find the values of all pronumerals (all external lines are tangents to the circles) in the circles below.

**a**

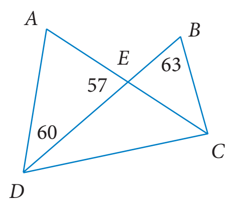



**b**



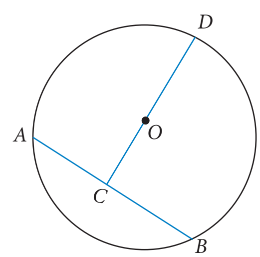

## Reasoning and communication

7 Example 12 Prove that $A$, $B$, $C$, $D$ are concyclic.

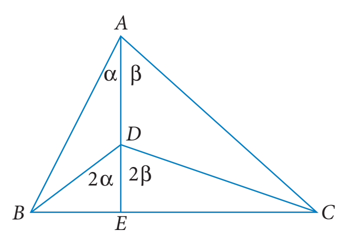



8 A circle with centre $O$ has radius $r$ and chord $AB = x$ with $\angle ACD = 90°$.

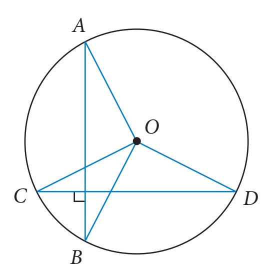
Show that $CD = \dfrac{2r + \sqrt{4r^2 - x^2}}{2}$.




9 In the triangles below, $\angle BDE = 2\angle BAD$ and $\angle CDE = 2\angle CAD$. Prove that a circle can be drawn through $A$, $B$ and $C$ with centre $D$.




10 Two chords $AB$ and $CD$ intersect at 90°. Prove, for obtuse $\angle AOD$, that $\angle AOD + \angle COB = 180°$, where $O$ is the centre of the circle.




11 Prove that if a pair of opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.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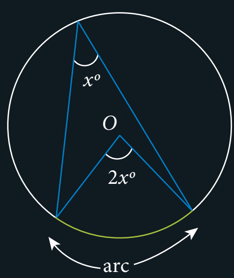
# CHAPTER SUMMARY
# CIRCLE GEOMETRY

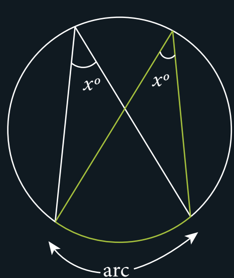
**Theorem 1: Angles at the centre of a circle**

- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc.




**Theorem 2: Angles on the same arc**

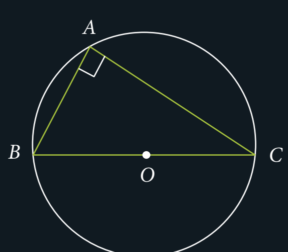
- Angles at the circumference of a circle subtended by the same arc are equal.




**Theorem 3: Semicircle angle**

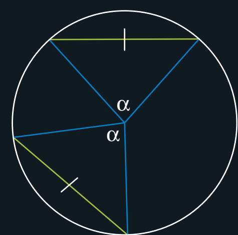
- An angle at the circumference subtended by a diameter is a right angle.

  Conversely, a chord that subtends a right angle at the circumference is a diameter.




**Theorem 4: Centre angles on equal chords**

- Equal chords of a circle subtend equal angles at the centre. Conversely, equal angles at the centre subtend equal chords.

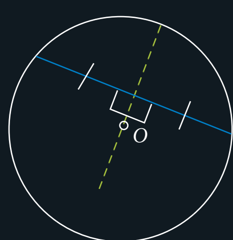



**Corollary**

Angles subtended at the centre or at the circumference by equal arcs are equal; conversely, equal angles at the centre subtend equal arcs.

**Theorem 5: Chord bisector**

- A radius that bisects a chord is perpendicular to the chord.

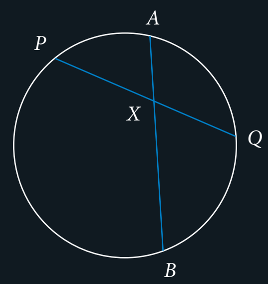



Corollaries are:

1. The perpendicular bisector of a chord passes through the centre of a circle.
2. The line passing through the centre of a circle perpendicular to a chord bisects the chord.
3. The perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.
4. Equal chords are equidistant from the centre of a circle.

**Theorem 6: Intersecting chords**

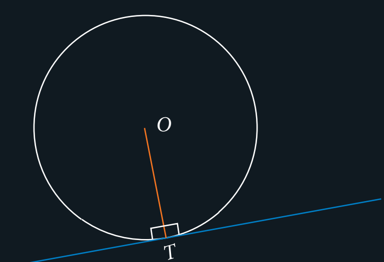
- When two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord.




In the circle, $AX \cdot BX = PX \cdot QX$

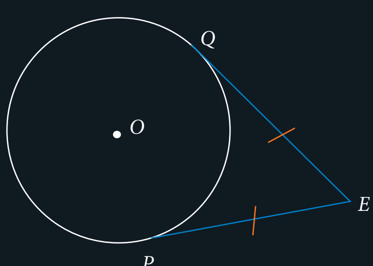
**Theorem 7: Tangent and radius**

- A tangent drawn to a circle is perpendicular to the radius at the point of contact.



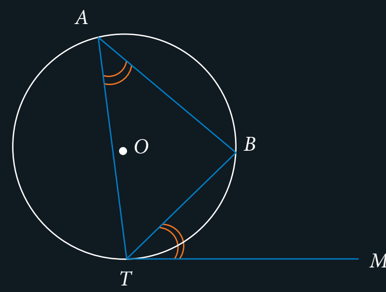

**Theorem 8: External tangents**

- Tangents drawn to a circle from an external point are equal in length.




**Theorem 9: Alternate segment**

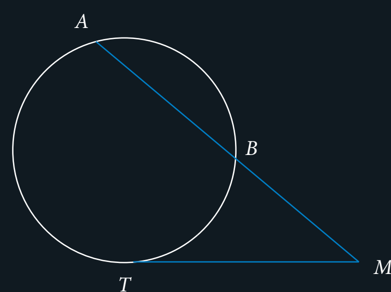
- The angle between the tangent and a chord is equal to the angle in the alternate segment.




In the diagram, $\angle TAB = \angle BTM$

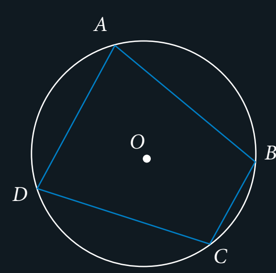
**Theorem 10: Tangent and secant**

- When a secant and a tangent are drawn to a circle from an external point, the square of the length of the tangent equals the product of the lengths to the circle on the secant.




In the diagram, $AM \cdot BM = TM^2$

- A **cyclic quadrilateral** is a quadrilateral whose four vertices lie on the circumference of a circle.




In the diagram, $ABCD$ is a cyclic quadrilateral.



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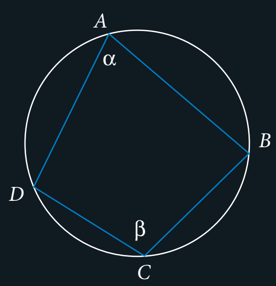
**Theorem 11: Opposite angles in a cyclic quadrilateral**

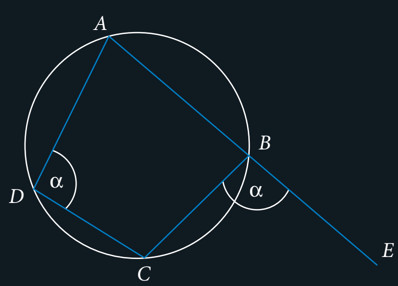
- The opposite angles of a cyclic quadrilateral are supplementary.



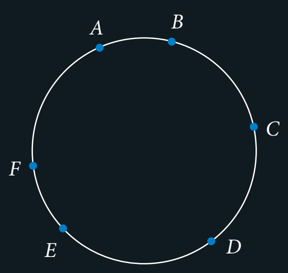

In the diagram, $\alpha + \beta = 180°$

**Theorem 12: Exterior angle of a cyclic quadrilateral**

- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



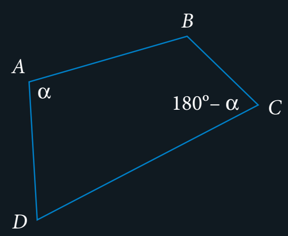

- Four or more points are called **concyclic** if they all lie on the same circle.




In the diagram, *A*, *B*, *C*, *D*, *E*, *F* are concyclic.

**Converse of Theorem 11**

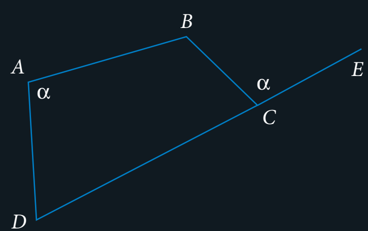
- If the opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.




In the diagram, *ABCD* is a cyclic quadrilateral. The four vertices *A, B, C, D* are concyclic.

**Converse of Theorem 12**

- If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.



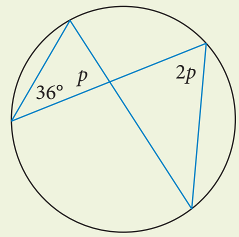

In the diagram, *ABCD* is a cyclic quadrilateral.

# 6 CHAPTER REVIEW
## CIRCLE GEOMETRY

### Multiple choice

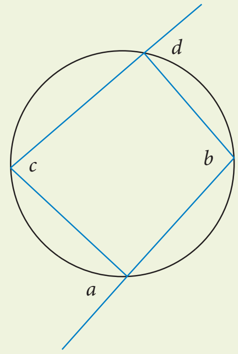
1 Example 2 In the diagram, $p$ = ...

A 18° B 36°
C 48° D 60°
E 72°



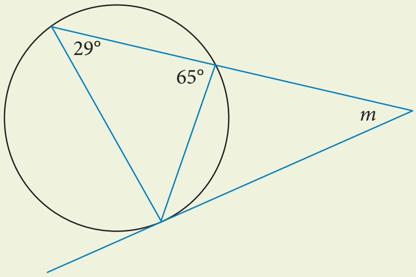

2 Example 9 In the diagram which statement is always true?

A $d = b$ B $b = c$
C $b = c$ D $d + c = 180°$
E $b + c = 180°$



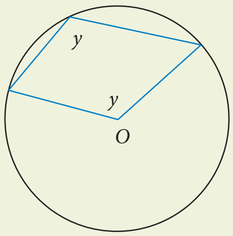

3 Example 7 In the diagram $m$ = ...

A 29° B 36°
C 61° D 86°
E 94°

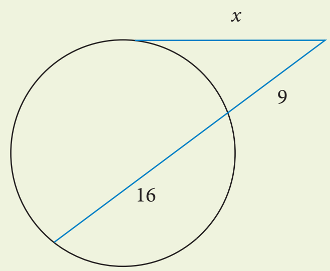



4 Example 1 In the diagram $y$ = ...

A 90° B 100°
C 110° D 120°
E 150°

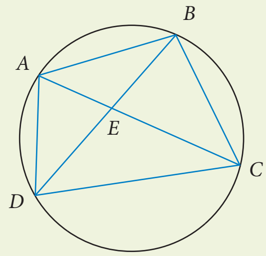


5 Example 8 In the diagram $x = ...$




A 5
B 9
C 10
D 12
E 15

6 Example 2 In the diagram, consider the statements

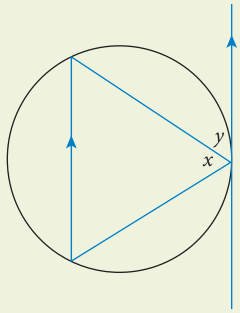



1 $\triangle AED \,|||\, \triangle BEC$
2 $\triangle ACD \,|||\, \triangle BDC$

Which of the following are true?

A both 1 and 2
B neither 1 nor 2
C only 1
D only 2
E It is impossible to determine

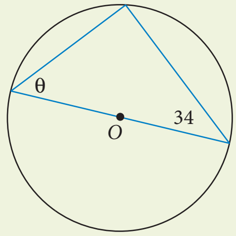
7 Example 8 In the diagram,



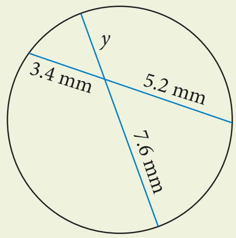

A $x = y$
B $x = 2y$
C $x = \dfrac{180° - y}{2}$
D $x = 180° - 2y$
E $x = 120° - 2y$

## Short answer

8 Example 3 $O$ is the centre of the circle. Evaluate $\theta$.




9 Example 6 Evaluate $y$, correct to 1 decimal place.

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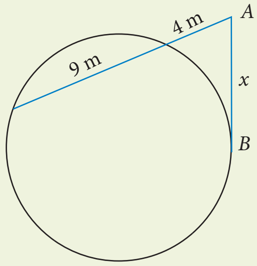
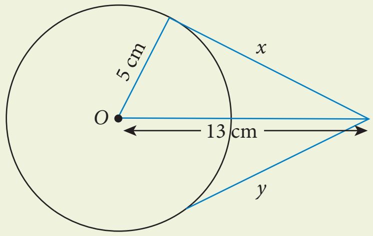
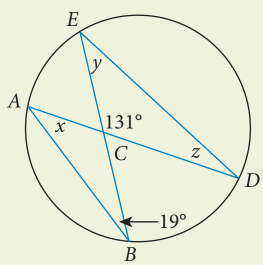
10 Example 8 *AB* is a tangent to the circle. Find the value of $x$ correct to 1 decimal place.



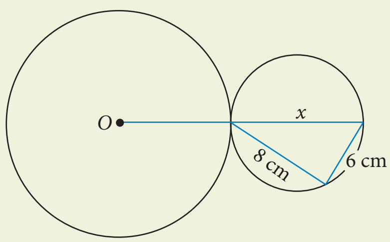

11 Example 7 *O* is the centre of the circle. Find the length of tangents $x$ and $y$.




12 Example 2 Evaluate $x$, $y$ and $z$, giving reasons for each step of your working.




13 Example 3 *O* is the centre of the larger circle. Find the value of $x$.



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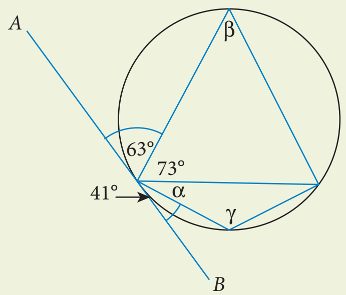
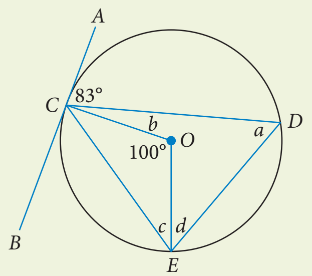
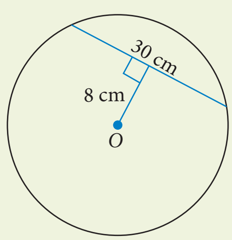
**14** Examples 8,9 *AB* is a tangent to the circle. Evaluate $\alpha$, $\beta$ and $\gamma$.



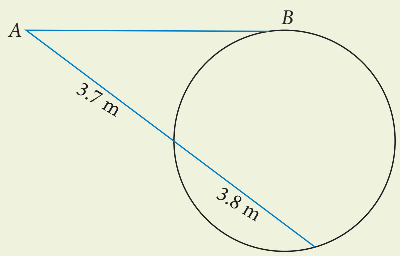

**15** Examples 1, 7 *O* is the centre of the circle, and *AB* is a tangent. Evaluate *a*, *b*, *c* and *d*, giving reasons for each step of your working.



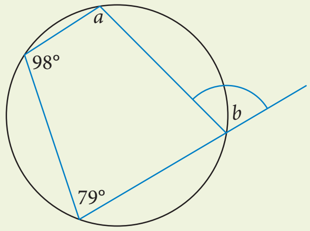

**16** Example 5 Find the length of the radius of the circle. *O* is the centre.



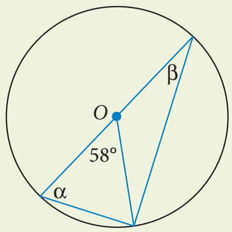

**17** Example 8 Find the length of tangent *AB*.



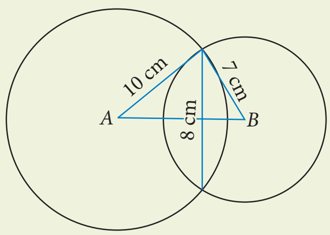
18 Example 9 Evaluate $a$ and $b$.



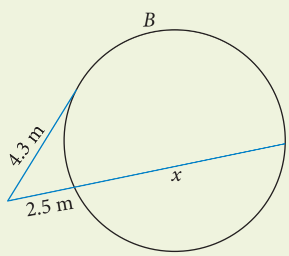

19 Examples 1, 3 $O$ is the centre of the circle. Find the value of $\alpha$ and $\beta$.



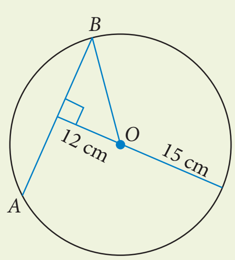

20 Example 11 Calculate the length of $AB$, correct to 3 significant figures, given that $A$ and $B$ are the centres of the circles.




21 Example 8 Find the value of $x$ correct to 1 decimal place.




22 Example 5 Find the length of $AB$.



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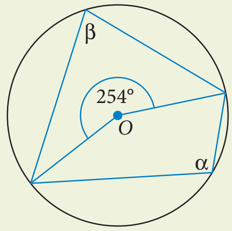
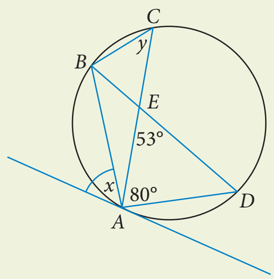
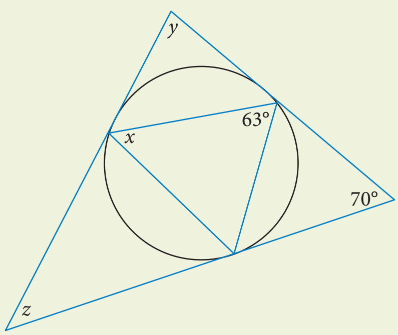
23 Example 9 Evaluate α and β.




24 Examples 2, 8 Evaluate $x$ and $y$, giving reasons for your working.

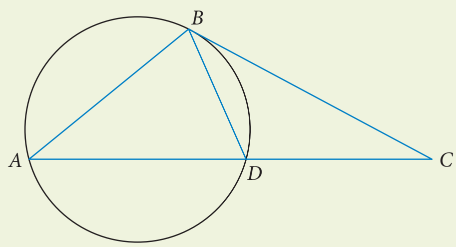



25 Example 7 Evaluate $x$, $y$ and $z$.




## Application

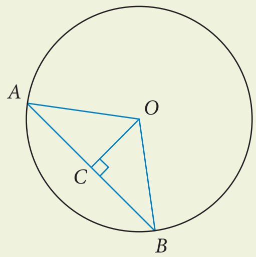
26 Prove that $\triangle BCD$ is similar to $\triangle ABC$.



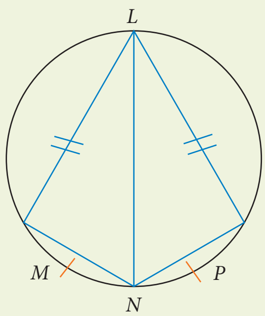
**27** $O$ is the centre of the circle.

**a** Prove that $\triangle OAC$ and $\triangle OBC$ are congruent.

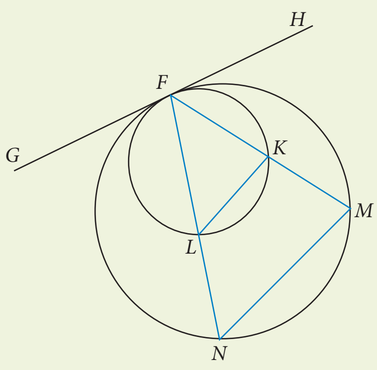
**b** Show that $OC$ bisects $AB$.




**28** In the circle, arc $MN$ = arc $PN$, $LM = LP$. Prove that the quadrilateral $LMNP$ is a kite.

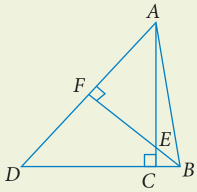



**29** $GFH$ is a common tangent to both circles. Prove that $LK \parallel NM$.




**30** In the triangle $DAB$, $AC$ and $BF$ are altitudes. Prove that $D$, $C$, $E$, $F$ are concyclic.

Practice quiz



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